The spectral problem of the modular oscillator in the strongly coupled regime

Rinat Kashaev

University of Geneva

Joint work with Sergey Sergeev (University of Canberra, Australia)

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Motivation: topological strings

$$\text{Toric CY 3-fold } M \xrightarrow{\text{Mirror Symmetry}} \boldsymbol{\rho}_M \quad \text{(trace class operator)}$$

The spectrum of $\boldsymbol{\rho}_M$ is expected to be related to enumerative invariants of M through the topological string partition functions. Suggested by Aganagic–Dijkgraaf–Klemm–Mariño–Vafa (2006) and materialized by Grassi–Hatsuda–Mariño (2016).

Example: the local $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_0$

$$\boldsymbol{\rho}^{-1} = \boldsymbol{\rho}_{\mathbb{F}_0,m}^{-1} = \boldsymbol{v} + \boldsymbol{v}^{-1} + \boldsymbol{u} + m\boldsymbol{u}^{-1}, \quad m \in \mathbb{R}_{>0},$$

with positive self-adjoint operators $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfying the Heisenberg–Weyl commutation relation $\boldsymbol{u}\boldsymbol{v} = e^{\mathrm{i}\hbar}\boldsymbol{v}\boldsymbol{u}$, $\hbar \in \mathbb{R}_{>0}$.

Implications of the Grassi–Hatsuda–Mariño conjecture

Fredholm determinant

$$\det(1+\kappa\boldsymbol{\rho}) = 1 + \sum_{N=1}^{\infty} Z(N,\hbar)\kappa^N \quad \text{(convergent series)}$$

where the fermionic spectral traces $Z(N,\hbar) = e^{F(N,\hbar)}$ provide a non-perturbative definition of the topological string partition functions.

$$\hbar \to \infty, \quad N \to \infty, \quad \text{with fixed } \lambda := \frac{\hbar}{N} \quad \text{(t'Hooft limit)}$$

$$F(N,\hbar) \simeq \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda)\hbar^{2-2g} \quad \text{(asymptotic series)}$$

with the genus g standard topological string free energies $\mathcal{F}_g(\lambda)$ in the conifold frame where λ is a flat coordinate for the CY moduli space vanishing at the conifold point.

Statement of the problem

For $\mathsf{b} \in \mathbb{C}_{\neq 0}$, define operators in $L^2(\mathbb{R})$

$$\boldsymbol{u} := \mathrm{e}^{2\pi \mathsf{b} \boldsymbol{x}}, \quad \boldsymbol{v} := \mathrm{e}^{2\pi \mathsf{b} \boldsymbol{p}}, \quad \bar{\boldsymbol{u}} := \mathrm{e}^{2\pi \mathsf{b}^{-1} \boldsymbol{x}}, \quad \bar{\boldsymbol{v}} := \mathrm{e}^{2\pi \mathsf{b}^{-1} \boldsymbol{p}}.$$

with Heisenberg operators

$$\boldsymbol{x}\psi(x) = x\psi(x), \quad \boldsymbol{p}\psi(x) = (2\pi \mathrm{i})^{-1}\psi'(x).$$

Spectral problem for two Hamiltonians

$$\boldsymbol{H} := \boldsymbol{v} + \boldsymbol{v}^{-1} + \boldsymbol{u} + \boldsymbol{u}^{-1}, \quad \bar{\boldsymbol{H}} := \bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}^{-1} + \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}^{-1}$$

which (formally) commute (*Faddeev's modular duality*).
Strongly coupled regime

$$\mathsf{b} = \mathrm{e}^{\mathrm{i}\theta},\ 0 < \theta < \frac{\pi}{2} \Rightarrow \bar{\boldsymbol{H}} = \boldsymbol{H}^* \quad \text{(Hermitian conjugate).}$$

Small b limit

$$\boldsymbol{H} = 4 + (2\pi\mathsf{b})^2(\boldsymbol{p}^2 + \boldsymbol{x}^2) + \mathcal{O}(\mathsf{b}^4) \quad \text{(“modular oscillator”).}$$

Functional difference equations

The common spectral problem for $\boldsymbol{H}$ and $\bar{\boldsymbol{H}}$ is equivalent to constructing an element $\psi(x) \in L^2(\mathbb{R})$ admitting analytic continuation to a domain containing the strip $|\Im z| \leq \max(\Re \mathsf{b}, \Re \mathsf{b}^{-1})$, satisfying the functional difference equations

$$\psi(x + \mathrm{i}\mathsf{b}) + \psi(x - \mathrm{i}\mathsf{b}) = (\varepsilon - 2\cosh(2\pi \mathsf{b} x))\psi(x),$$

$$\psi(x + \mathrm{i}\mathsf{b}^{-1}) + \psi(x - \mathrm{i}\mathsf{b}^{-1}) = (\bar{\varepsilon} - 2\cosh(2\pi \mathsf{b}^{-1} x))\psi(x),$$

and the restrictions $\psi(x + \mathrm{i}\lambda)$ being elements of $L^2(\mathbb{R})$, where $\lambda \in \{\Re \mathsf{b}, -\Re \mathsf{b}, \Re \mathsf{b}^{-1}, -\Re \mathsf{b}^{-1}\}$.

In the general case of Baxter's $T - Q$ equations, an approach for constructing the solution in the strongly coupled regime is suggested by S. Sergeev (2005).

A different approach through auxiliary non-linear integral equations is developed by O. Babelon, K. Kozlowski, V. Pasquier (2018).

Behavior at infinity

In the limit $x \to -\infty$, equation

$$\psi(x + \mathrm{ib}) + \psi(x - \mathrm{ib}) = (\varepsilon - 2\cosh(2\pi \mathrm{b} x))\psi(x)$$

is approximated by the equation

$$\psi(x + \mathrm{ib}) + \psi(x - \mathrm{ib}) = -\,\mathrm{e}^{-2\pi \mathrm{b} x}\,\psi(x),$$

where, in the left hand side, any one of the two terms can be dominating giving rise to two possible asymtotics

$$\psi(x)|_{x\to-\infty} \sim \mathrm{e}^{\pm \mathrm{i}\pi x^2 + 2\pi\eta x}, \quad \eta := \frac{\mathrm{b} + \mathrm{b}^{-1}}{2} = \cos\theta.$$

Thus, there are two solutions of the form

$$\psi_\pm(x) = \mathrm{e}^{\pm \mathrm{i}\pi x^2 + 2\pi\eta x}\,\phi_\pm(x), \quad \phi_\pm(x)|_{x\to-\infty} = \mathcal{O}(1).$$

Thus, a general exponentially decaying at $x \to -\infty$ solution is of the form

$$\psi(x) = \mathrm{e}^{2\pi\eta x}\left(\mathrm{e}^{\mathrm{i}\pi x^2}\,\phi_+(x) + \mathrm{e}^{-\mathrm{i}\pi x^2}\,\phi_-(x)\right).$$

The factorization ansatz

We look for solutions of the form

$$\psi_\pm(x) = \mathrm{e}^{\pm \mathrm{i}\pi x^2 + 2\pi\eta x}\,\phi_\pm(x)$$

with

$$\phi_+(x) = f\left(e^{\pi \mathrm{i} b^2}, \varepsilon, e^{2\pi \mathrm{b} x}\right)\overline{f\left(e^{-\pi \mathrm{i} b^2}, \varepsilon, e^{2\pi \mathrm{b} x}\right)}, \quad \phi_-(x) = \alpha\overline{\phi_+(x)},$$

where $\alpha \in \mathbb{C}$ and

$$f(q, \varepsilon, u) = \sum_{n=0}^{\infty} c_n(q, \varepsilon) u^n$$

solves the functional equation

$$f\left(q, \varepsilon, uq^{-2}\right) + q^2 u^2 f(q, \varepsilon, q^2 u) = (1 - \varepsilon u + u^2) f(q, \varepsilon, u).$$

The main functional equation

$$f\left(u/q^2\right) + q^2u^2f(q^2u) = (1-\varepsilon u + u^2)f(u), \quad q := e^{\pi i \mathsf{b}^2}.$$

Involution in the space of solutions: $f(u) \mapsto \check{f}(u) := u^{-1}f\left(u^{-1}\right)$.
An equivalent first order difference matrix equation

$$\begin{pmatrix} f\left(u/q^2\right) \\ f(u) \end{pmatrix} = L(u)\begin{pmatrix} f(u) \\ f(q^2u) \end{pmatrix}, \; L(u) := \begin{pmatrix} 1-\varepsilon u + u^2 & -q^2u^2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} f\left(u/q^2\right) \\ f(u) \end{pmatrix} = M_n(u)\begin{pmatrix} f\left(q^{2n-2}u\right) \\ f(q^{2n}u) \end{pmatrix}, \quad \forall n \in \mathbb{Z}_{>0},$$

$$M_n(u) := L(u)L(q^2u)\cdots L(q^{2(n-1)}u), \quad M_\infty(u) = \begin{pmatrix} \chi_q\left(u/q^2\right) & 0 \\ \chi_q(u) & 0 \end{pmatrix},$$

where $\chi_q(u) = \chi_q(u,\varepsilon)$ is an entire function of $u \in \mathbb{C}$ normalised so that $\chi_q(0) = 1$ and which solves the main functional equation. The second solution $\check{\chi}_q(u) := u^{-1}\chi_q(u^{-1})$ leads to a non-zero Wronskian $[\chi_q, \check{\chi}_q](u) := \chi_q\left(q^{-2}u\right)\check{\chi}_q(u) - \check{\chi}_q\left(q^{-2}u\right)\chi_q(u)$.

Orthogonal polynomials associated to $\boldsymbol{\chi}_q(u,\varepsilon)$

$$\boldsymbol{\chi}_q(u,\varepsilon)=\sum_{n\geq 0}\frac{\chi_{q,n}(\varepsilon)}{(q^{-2};q^{-2})_n}u^n=\sum_{n\geq 0}(-1)^n q^{n(n+1)}\frac{\chi_{q,n}(\varepsilon)}{(q^2;q^2)_n}u^n\,.$$

with polynomials $\chi_{q,n}(\varepsilon)\in\mathbb{C}[\varepsilon]$ satisfying the *recurrence relation*

$$\chi_{q,0}(\varepsilon)=1\,,\quad \chi_{q,n+1}(\varepsilon)=\varepsilon\chi_{q,n}(\varepsilon)+(q^n-q^{-n})^2\chi_{q,n-1}(\varepsilon),$$

with few first polynomials

$$\chi_{q,1}(\varepsilon)=\varepsilon,\quad \chi_{q,2}(\varepsilon)=\varepsilon^2+(q-q^{-1})^2,$$
$$\chi_{q,3}(\varepsilon)=\varepsilon(\varepsilon^2+(q^2-q^{-2})^2+(q-q^{-1})^2),\quad \dots$$

Multiplication rule

$$\chi_{q,m}(\varepsilon)\chi_{q,n}(\varepsilon)$$
$$=\sum_{k=0}^{\min(m,n)}\frac{(q^{2m};q^{-2})_k(q^{2n};q^{-2})_k(q^{2(k-m-n)};q^2)_k}{(q^2;q^2)_k}\chi_{q,m+n-2k}(\varepsilon)$$

The main functional equation with q replaced by q^{-1}

$$f(q^2u) + \frac{u^2}{q^2} f\left(\frac{u}{q^2}\right) = (1 - \varepsilon u + u^2) f(u).$$

There is no solution regular at $u = 0$. The series

$$\boldsymbol{\chi}_{q^{-1}}(u, \varepsilon) \simeq \sum_{n \geq 0} \frac{\chi_{q,n}(\varepsilon)}{(q^2; q^2)_n} u^n$$

does not converge, it is only an asymptotic expansion of the true solution

$$\boldsymbol{\chi}_{q^{-1}}(u, \varepsilon) := \frac{\check{\boldsymbol{\chi}}_q(u, \varepsilon)}{[\boldsymbol{\chi}_q, \check{\boldsymbol{\chi}}_q](u)}.$$

Result for the eigenfunction

$$\psi(x) := \mathsf{b}^{-1}\,\mathsf{e}^{\pi \mathsf{i}\sigma^2-\xi\pi\mathsf{i}/4}\,\mathsf{e}^{2\pi\eta x+\mathsf{i}\pi x^2}\,\frac{\check{\boldsymbol{\chi}}_q(u)\overline{\boldsymbol{\chi}_q(u)}+\xi\boldsymbol{\chi}_q(u)\overline{\check{\boldsymbol{\chi}}_q(u)}}{\theta_1(su,q)\theta_1(s^{-1}u,q)}.$$

where

$$[\boldsymbol{\chi}_q,\check{\boldsymbol{\chi}}_q](u)=\varrho\theta_1(su,q)\theta_1(s^{-1}u,q),$$

$$\theta_1(u,q) := \frac{1}{\mathsf{i}}\sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2}u^{n+1/2},$$

with certain functions $s=s(\varepsilon,q)$, $\varrho=\varrho(\varepsilon,q)$, $s:=\mathsf{e}^{2\pi\mathsf{b}\sigma}$, and the variable $\xi\in\{\pm1\}$ is the *parity* of the eigenstate: $\psi(-x)=\xi\psi(x)$. The function is real $\overline{\psi(x)}=\psi(x)$ (thus modular invariant $\mathsf{b}\leftrightarrow\mathsf{b}^{-1}$) and exponentially decays at both infinities

$$|\psi(x)|\sim \mathsf{e}^{-2\pi\eta|x|},\quad x\to\pm\infty.$$

Quantization condition for the eigenvalues

The quantization condition is the analyticity condition for $\psi(x)$ with complex x in the strip

$$S_{\mathsf{b}} := \left\{ z \in \mathbb{C} \mid |\Im z| < \max(|\Re \mathsf{b}|, |\Re \mathsf{b}^{-1}|) \right\}.$$

Define

$$G_q(u,\varepsilon) := \frac{\chi_q(u,\varepsilon)}{\check{\chi}_q(u,\varepsilon)}, \quad G_q(u,\varepsilon)G_q(1/u,\varepsilon) = 1, \quad \forall u \in \mathbb{C}_{\neq 0}.$$

Theorem

Let $\varepsilon = \varepsilon(\sigma)$ be such that $[\chi_q, \check{\chi}_q](u) = \varrho\theta_1(su)\theta_1(s^{-1}u)$ for any $u \in \mathbb{C}$, and assume that $s \notin \pm q^{\mathbb{Z}}$ (recall that $s = s(\sigma) = \mathrm{e}^{2\pi \mathsf{b}\sigma}$). Then the eigenfunction $\psi(x)$ does not have poles in the strip S_{b} if the variable σ is such that $G_q(s,\varepsilon) = -\xi\overline{G_q(s,\varepsilon)}$. Moreover, in that case, $\psi(x)$ is an entire function on $\mathbb{C}$.