3d Modularity

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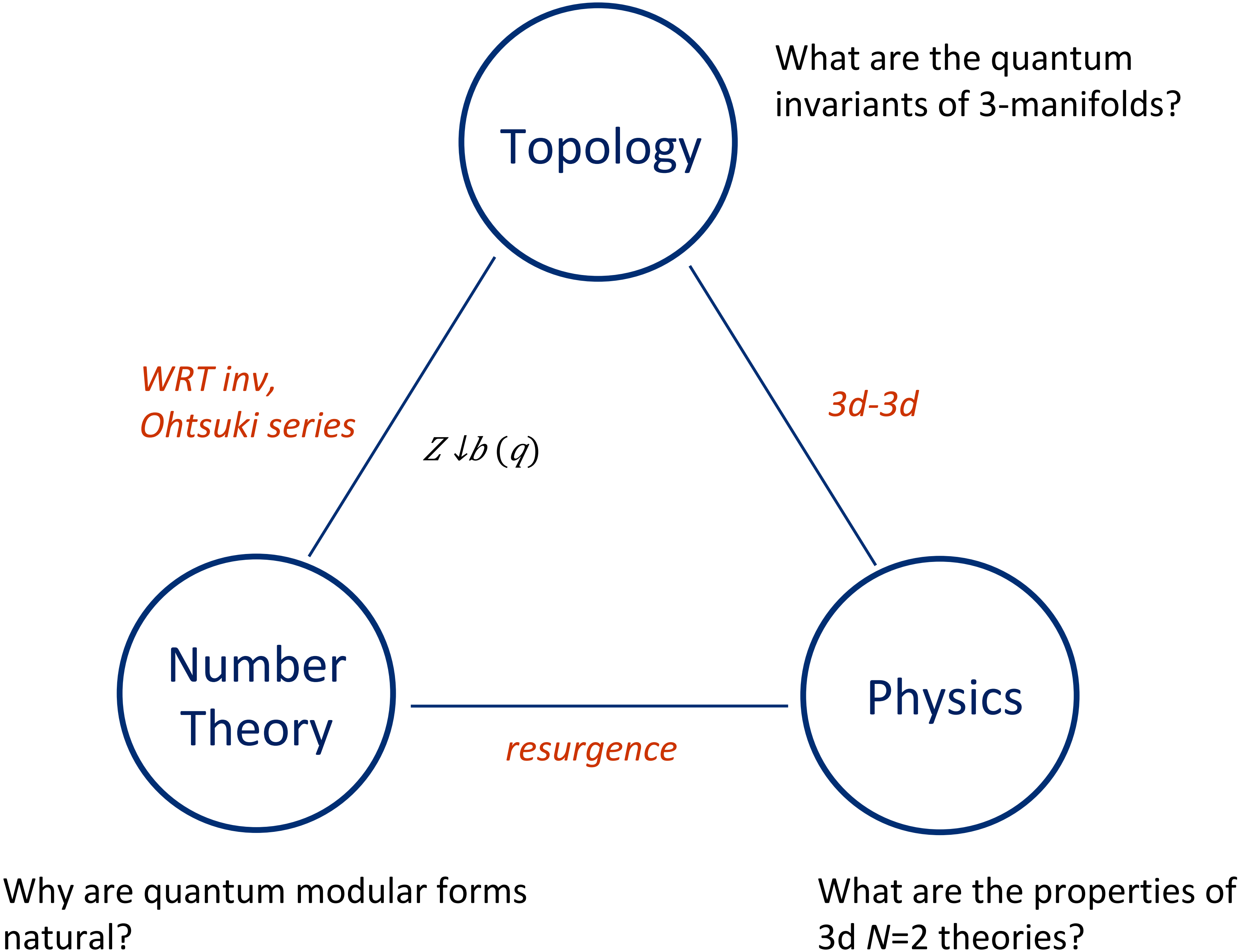

What are the quantum invariants of 3-manifolds?
Topology
WRT inv,
Ohtsuki series
Z ↓b (q)
3d-3d
Number Theory
resurgence
Physics
Why are quantum modular forms natural?
What are the properties of 3d N=2 theories?

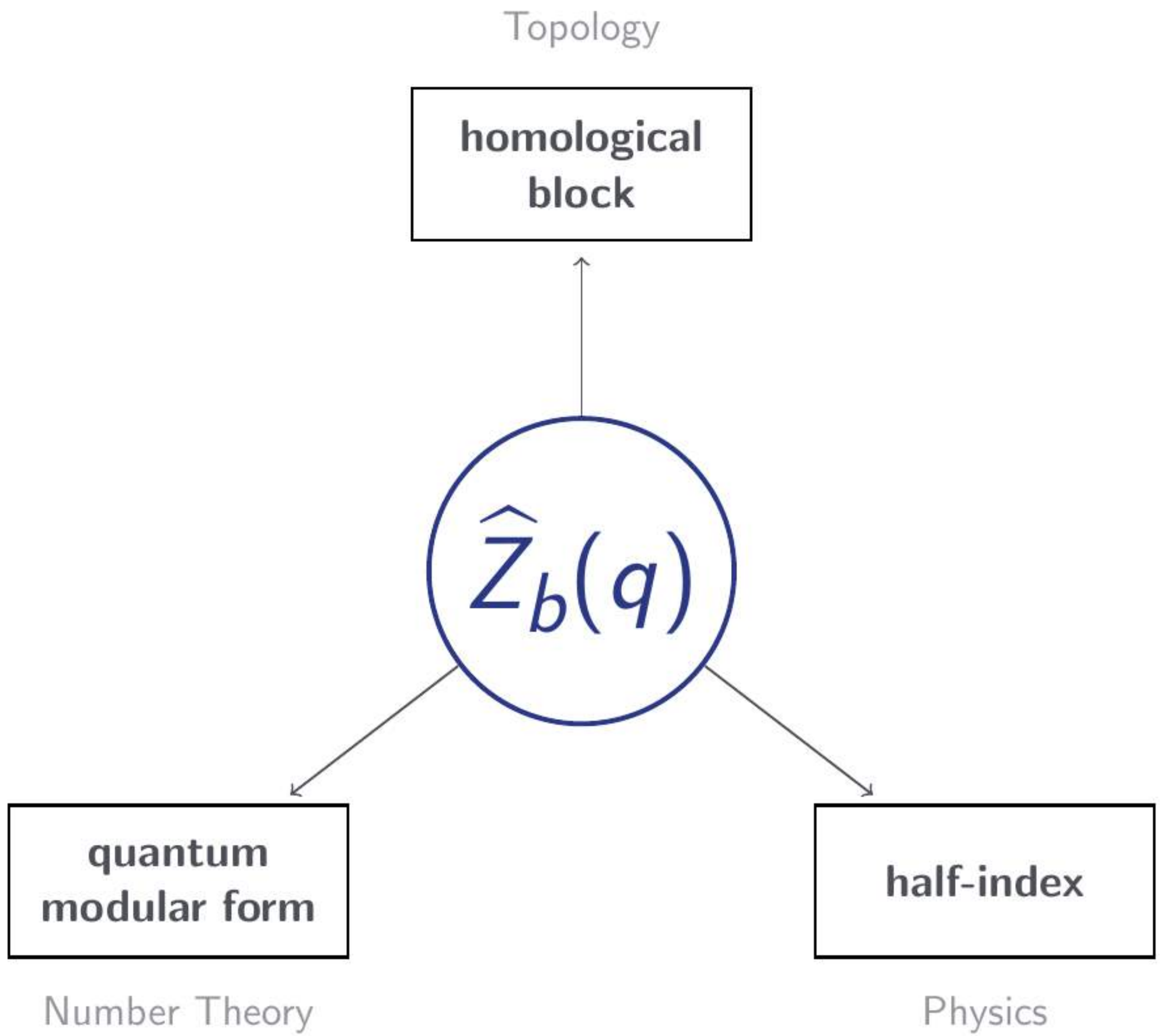
Outline

I. What is $\widehat{Z}_b(q)$?

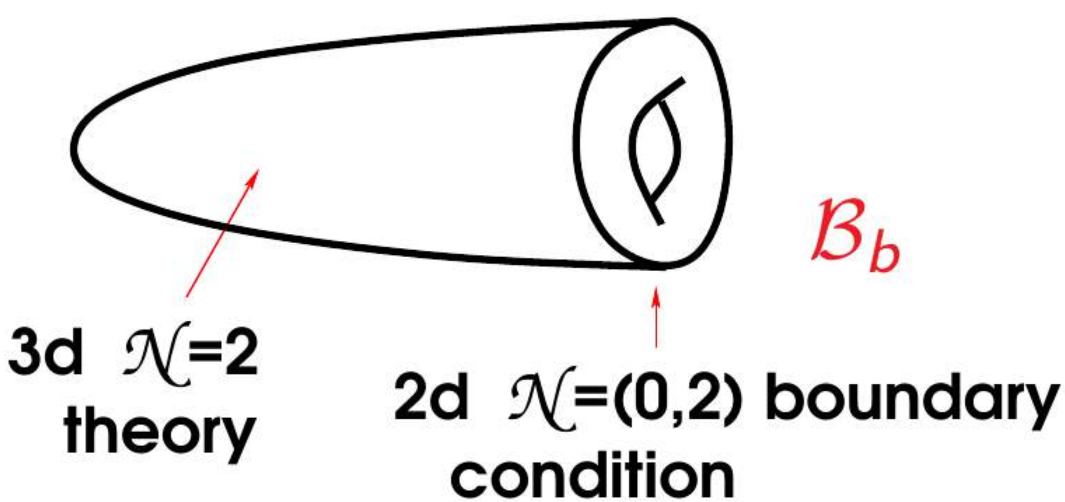
II. What are *False Theta Functions*?

III. The *False-Mock Pair* and *Quantum Modular Forms*

I. What is $\widehat{Z}_b(q)$?



Bulk 3d Coupled to Boundary 2d Systems

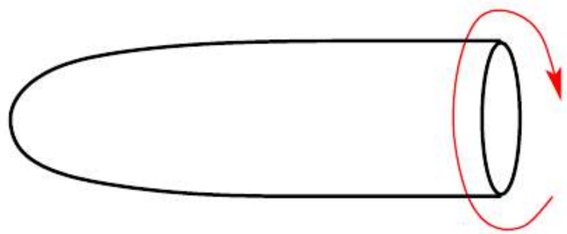


$$\widehat{Z}_b(q) \; := \; Z_{\mathcal{T}}(D^2 \times_q S^1; \mathcal{B}_b)$$

$$= \textbf{“Half-Index”}$$

$$(q = e^{2\pi i \tau})$$

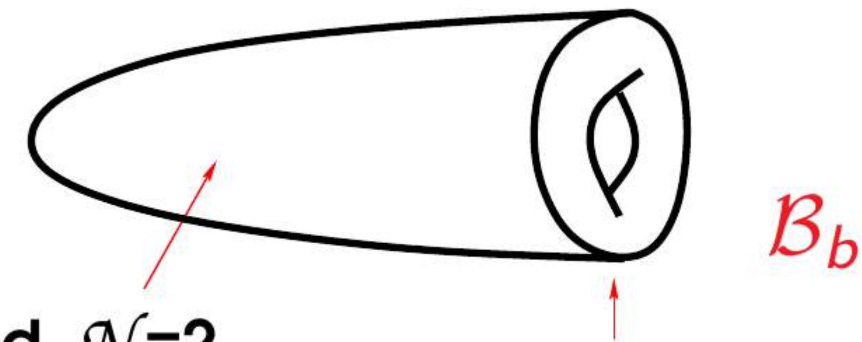
Counts BPS States

(time) x 

$$\widehat{Z}_b(q) = \textbf{"Half-Index"}$$

$$\in q^{\Delta}\mathbb{Z}[[q]]$$

Localisation $\Rightarrow$ Contour integral



$$\widehat{Z}_b(q) := Z_{\mathcal{T}}(D^2 \times_q S^1; \mathcal{B}_b)$$
$$= \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta_{2d,(b)}(x;q)$$

$$\widehat{Z}_b(q) = \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta_{2d,(b)}(x; q)$$

bulk theory	$\widehat{Z}_b(q)$	
gapped and trivial	modular	
gapped but in top. non-trivial phases	modified modularity	✓ **this talk**
not gapped	??	

6d (2,0) G-theory on $Cigar \times \mathbb{R}_t \times M_3$

3d $\mathcal{N} = 2$ thy $\mathcal{T}_G[M_3]$

SUSY vacua

B.C. $\mathcal{B}_b$

M_3 Topology

$G_{\mathbb{C}}$ flat connections

Ab. G **flat connections**

[Gukov–Pei–Putrov–Vafa, 17]

6d (2,0) G-theory on $Cigar \times \mathbb{R}_t \times M_3$

3d $\mathcal{N} = 2$ thy $\mathcal{T}_G[M_3]$

SUSY vacua

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M_3 Topology

$G_{\mathbb{C}}$ flat connections

Ab. G flat connections

$SU(2)$

$$\widehat{Z}_b(q) \text{ with } b \in (\mathrm{Tor} H_1(M_3, \mathbb{Z}))^* / \mathbb{Z}_2 \cong \pi_0 \mathcal{M}_{\mathsf{flat}}^{\mathsf{Ab.}}$$

$$\widehat{Z}_b(q) \xrightarrow[\text{summed over}]{\text{radial limit}} Z_{\text{CS}}$$

WRT inv.

$$\underline{Z_{\text{CS}}(M_3;k)} = \sum_a e^{2\pi i k \text{CS}(a)} \left(\lim_{q \to e^{2\pi i/k}} \sum_b S_{ab}^{(A)} \widehat{Z}_b(q) \right)$$

CS level

summing over b.c./Ab. flat connections

"Abelian S-matrix"

Z_a

transseries expansion

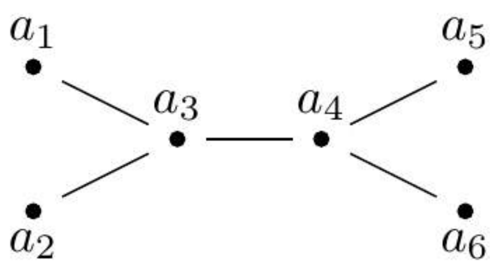
$$\sim \sum_\alpha e^{2\pi i k \text{CS}(\alpha)} \underline{Z_\alpha^{\text{pert}}(k)}$$

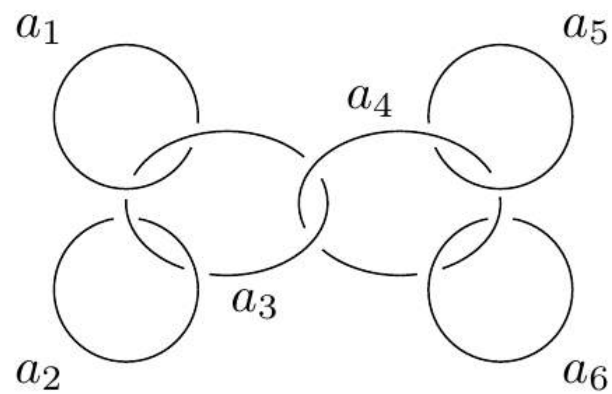
Ohtsuki series ($\alpha = 0$)

summing over all $SU(2)$ flat connections

[Gukov–Pei–Putrov–Vafa, 17]

Plumbed Three-Manifolds





↓

Adjacency Matrix

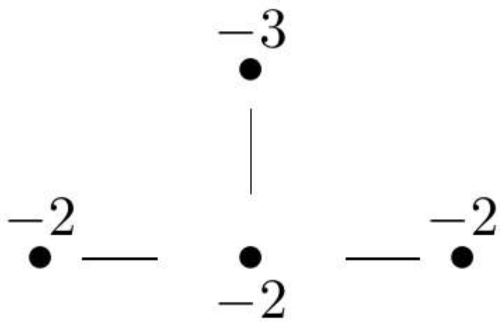
↓ for a large fam.

$\widehat{Z}_b(M_3)$

↓

Plumbed M_3 inc. all Seifert man.

eg.



$$\widehat{Z}_1(M_3;q) = q^{-\frac{3}{8}}(1 - q + q^2 + \dots) = q^{-\frac{5}{12}}(\Psi_{6,1} - \Psi_{6,5})(\tau)$$

$$\widehat{Z}_2(M_3;q) = q^{-\frac{1}{4}}(1 + q^4 + \dots) = q^{-\frac{5}{12}}\Psi_{6,2}(\tau)$$

previous related works [Lawrence–Zagier 99, Hikami 05-06]

II. What are false theta functions?

$$\widehat{Z}_b(q) \sim \Psi_{m,r}$$

??

$$\Psi_{m,r}(\tau) = \sum_{\substack{\ell \in \mathbb{Z} \\ \ell = r \bmod 2m}} \underline{\mathrm{sgn}(\ell)}\, q^{\ell^2/4m}$$

"false"

Relation to modular forms:

weight w mod. form $\xrightarrow[\text{integral}]{\text{Eichler}}$ another q series

$$g = \sum_{n>0} a_g(n) q^n \quad \xrightarrow[\text{integral}]{\text{Eichler}} \quad \widetilde{g}(\tau) := \sum_{n>0} n^{1-w} a_g(n) q^n$$

(usual θ-function)

$$\theta_{m,r}(\tau,z) = \sum_{\ell = r \bmod 2m} q^{\ell^2/4m} y^\ell$$

$$\Big\downarrow \frac{\partial}{\partial z}(\dots)|_{z=0}$$

$$\theta^1_{m,r}(\tau) = \sum_{\ell = r \bmod 2m} \ell\, q^{\ell^2/4m} \xrightarrow[\text{integral}]{\text{Eichler}} \Psi_{m,r} = \widetilde{\theta^1_{m,r}}$$

(weight 3/2 θ-function)

$\theta_m = (\theta_{m,r})$ $SL_2(\mathbb{Z})$ "Weil" representation

in general *not* irrep

Consider eigen-spaces of the orthogonal group:

$$O_m := \{a \in \mathbb{Z}/2m \mid a^2 = 1 \bmod (4m)\}$$

$$a : \theta_{m,r} \mapsto \theta_{m,r}$$

A useful label:

$$O_m \cong \mathrm{Ex}_m = \text{ group of exact divisors of } m$$

$$\cup$$

K : labels the $+1$ eigenspace

eg.

$$\Psi_1^{42+6,14,21}(\tau) = (\Psi_{42,1} - \Psi_{42,13} - \Psi_{42,29} + \Psi_{42,41})(\tau)$$
$$= q^{1/168}(1 - q - q^5 + O(q^{10}))$$

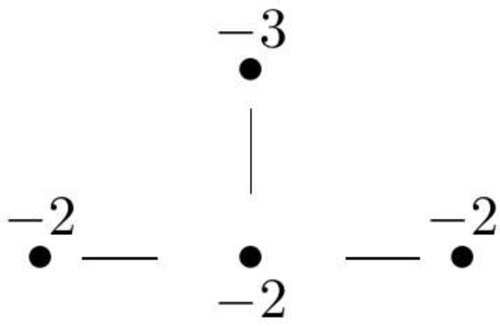
$m = 42, K = \{1, 6, 14, 21\}$, 3-dim irrep

$\widehat{Z}_0(M_3; q) \sim \Psi_1^{42+6,14,21}(\tau)$
for $M_3 = \Sigma(2,3,7)$ Brieskorn sphere .

Somehow 3-manifolds like irreps.

The other components of the vector will come to life as non-Ab flat conn.

eg.



Ψ_1^{6+2}

$$\widehat{Z}_1(M_3;q) = q^{-\frac{3}{8}}(1-q+q^2+\dots) = q^{-\frac{5}{12}}(\underline{\Psi_{6,1}-\Psi_{6,5}})(\tau)$$

$$\widehat{Z}_2(M_3;q) = q^{-\frac{1}{4}}(1+q^4+\dots) = q^{-\frac{5}{12}}\Psi_{6,2}(\tau)$$

Ψ_2^{6+2}

Resurgence: pert. asympt. series $\to$ an actual func. inc. non-pert.

Techniques: Borel resummation

$$Z_{\text{pert}}(k) = \sum_n \frac{a_n}{k^n} \quad \left(\frac{1}{k} = \hbar\right)$$

$$\downarrow$$

$$BZ_{\text{pert}}(z) = \sum_n \frac{a_n}{\Gamma(n)} z^{n-1}$$

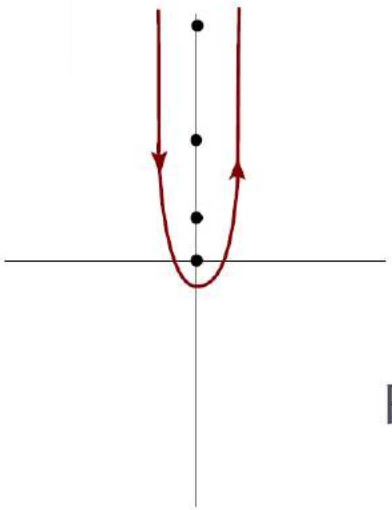
$$B\left(\frac{1}{\sqrt{k}}\Psi_{m,r}\left(\frac{1}{k}\right)\right)(z) = \frac{1}{\sqrt{\pi z}} \frac{\sin((m-r)\sqrt{\frac{2\pi z}{m}})}{\sin(m\sqrt{\frac{2\pi z}{m}})}$$

$$\downarrow$$

$$Z_{\text{tot}} = \int e^{-\tau z} BZ_{\text{pert}}(z) dz$$

$$\frac{1}{\sqrt{k}}\Psi_{m,r}\left(\frac{1}{k}\right) = \frac{\sqrt{i}}{2}\left(\int_{e^{i\delta}\mathbb{R}_+} + \int_{e^{-i\delta}\mathbb{R}_+}\right) \frac{dz}{\sqrt{\pi z}} \frac{\sin((m-r)\sqrt{\frac{2\pi z}{m}})}{\sin(m\sqrt{\frac{2\pi z}{m}})} e^{-ikz}$$

[Gukov–Marino–Putrov 16]



Res. of pole at $z = n^2$: $e^{-2\pi i\, k \frac{n^2}{4m}} S^{(W)}_{rn}$

Recall: $Z_{\mathrm{CS}}(M_3; k) = \sum_a e^{2\pi i k \mathrm{CS}(a)} \underbrace{\sum_b S^{(A)}_{ab} \widehat{Z}_b(q)}_{= Z_a}$

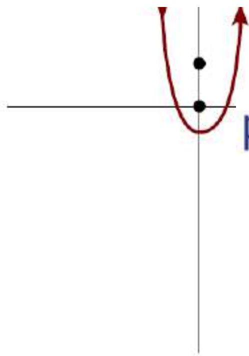
Consider $\widehat{Z}_b \sim \Psi^{m+K}_{r_b}$.

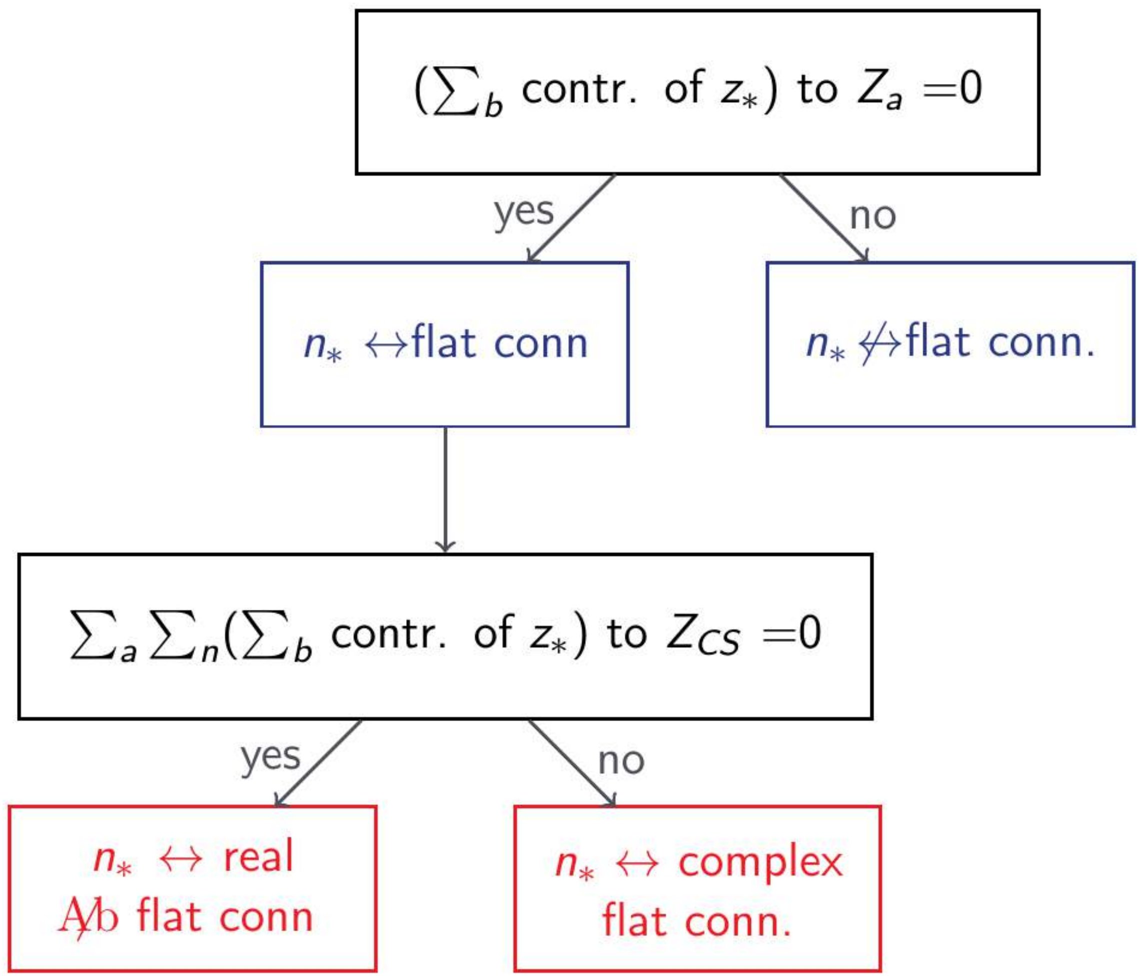
Contribution of the poles to Z_a : $\sum_b \sum_n e^{-2\pi i\, k \frac{n^2}{4m}} S^{(A)}_{ab} S^{(W)}_{r_b n}$

Poles are grouped into orbits of $K \subset O_m$.

Consider $\widehat{Z}_b \sim \Psi^{m+K}_{r_b}$.

Contribution of the poles to Z_a : $\sum_b \sum_n e^{-2\pi i\, k \frac{n^2}{4m}} S^{(A)}_{ab} S^{(W)}_{r_b n}$





From plumbing data to flat connections.

Compute $\widehat{Z}_b$

↓

Identify Weil rep. (m, K) with $\widehat{Z}_b = cq^{\delta}(\Psi_r^{m+K} + d)$

↓

Compute the S-matrices $S^{(A)}$ and $S^{(W)}$

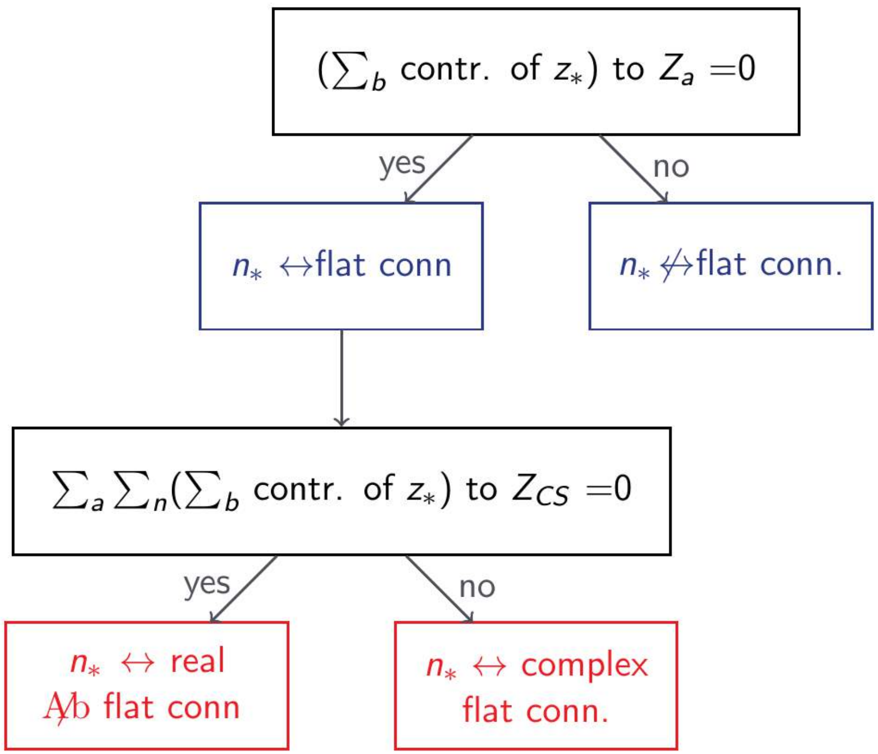
↓

apply the above

eg.

$\widehat{Z}_0(M_3; q) \sim \Psi_1^{42+6,14,21}(\tau)$
for $M_3 = \Sigma(2,3,7)$ Brieskorn sphere

3-dim $SL_2(\mathbb{Z})$ rep $\Rightarrow$ 3 groups of poles$\Rightarrow$ **3 non-Ab flat conn.**:



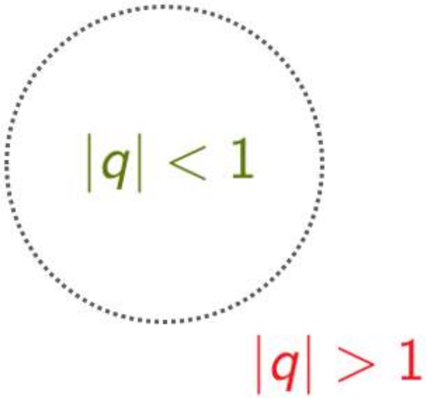
complex, $CS = -\frac{1^2}{4\times 42}$

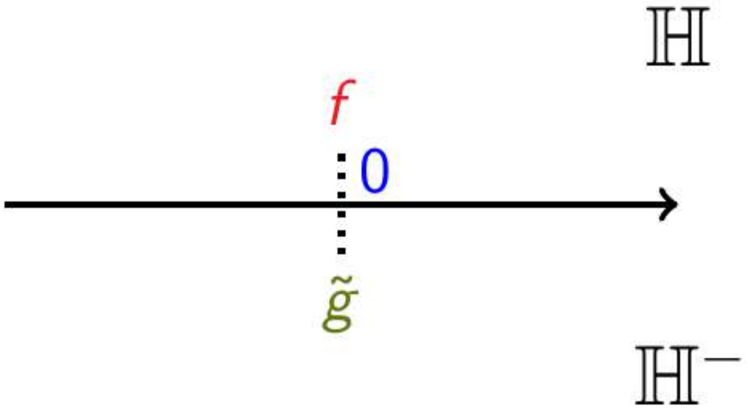
real, $CS = -\frac{5^2}{4\times 42}$

real, $CS = -\frac{11^2}{4\times 42}$

III. False, Mock, Quantum

[Zagier 10]

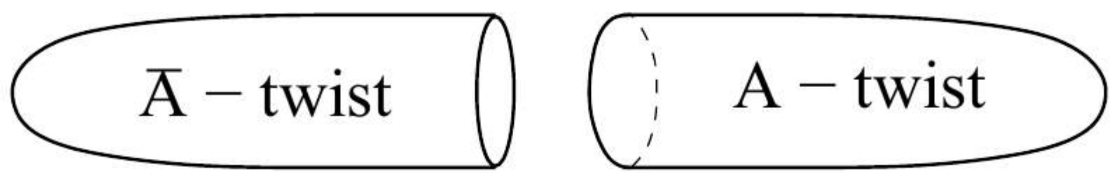




3d $\mathcal{N} = 2$ Superconformal Index

$$\mathcal{I}(q) := \mathrm{Tr}_{\mathcal{H}_{S^2}}(-1)^F q^{R/2+J_3}$$

$$= Z(S^2 \times_q S^1) \qquad \in \mathbb{Z}[[q]]$$



$$\sim \sum_b \widehat{Z}_b(q)\widehat{Z}_b(q^{-1})$$

[Gukov–Pei–Putrov–Vafa, 17]

What's this?
Re-expand! But how?

From CS:

$$Z_{\alpha}^{\mathrm{pert}}(M_3; k) = \sum_n a_n (\frac{1}{k})^n$$

$q \leftrightarrow q^{-1}$

$$\Leftrightarrow Z_{\alpha}^{\mathrm{pert}}(-M_3; k) = \sum_n a_n (-\frac{1}{k})^n$$

$$\Rightarrow \ \widehat{Z}_b(M_3; q^{-1}) \sim \widehat{Z}_b(-M_3; q)$$

What's this?

Some q-hypergeometric series:

$$\psi(q) := \frac{q^{\frac{1}{24}}}{2}\left(1 - \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(1+q)(1+q^2)\dots(1+q^n)}\right)$$

converges when $|q| < 1$ or $|q| > 1$

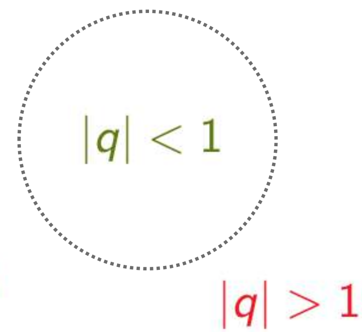
$$= \Psi_1^{6+2}(\tau) = q^{\frac{1}{24}}(1 - q + q^2 + \dots)$$

$$= \frac{q^{-\frac{1}{24}}}{2}\left(1 + q^{-1} - 2q^{-2} + 3q^{-3} + O(q^4)\right)$$

$$||$$

$$f_R(q^{-1})$$

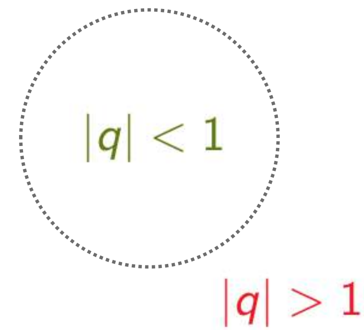
order 3 mock θ-func of Ramanujan



[Bringmann–Folsom–Rhoades 12]

III. False, Mock, Quantum: $q \leftrightarrow q^{-1}$/**mock**/quantum asymptotics

"I discovered very interesting functions recently which I call "Mock" theta functions. Unlike the "False" theta functions they enter into mathematics as beautifully as ordinary theta functions.",



Def:

$$\left\{ \begin{array}{c} f : \mathbb{H} \to \mathbb{C} \\ \hat{f} = f - g^* \\ g^*(\tau) := C \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-k} \overline{g(-\bar{\tau}')}\, d\tau' \\ g = \mathrm{shad}(f) \end{array} \right.$$

$f : \mathbb{H} \to \mathbb{C}$	holom., "**mock**"
$\hat{f} = f - g^*$	~~holom.~~, mod.
$g^*(\tau) := C \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-k} \overline{g(-\bar{\tau}')}\, d\tau'$	modular correction
$g = \mathrm{shad}(f)$	mod.form, "**shadow**"

f is mock θ if shad(f) is a θ-func.

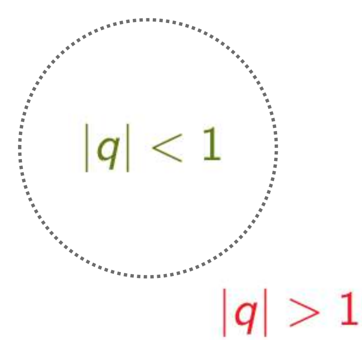
[Zwegers 02, ..., Zagier 07]

Example: a q-hypergeometric series

false: $\psi(q) = \Psi_1^{6+2}(\tau) = \widetilde{\theta_1^{6+2,1}(\tau)}$

$\uparrow$ shad$(f) = \theta_1^{6+2,1}$

mock: $\psi(q^{-1}) \sim f(q)$



Example: Rademacher sums

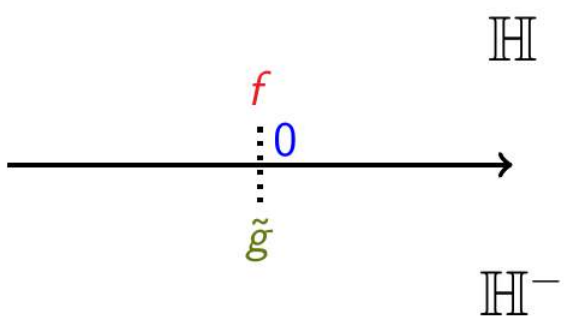
a regularised sum over $SL_2(\mathbb{Z})$ images

Theorem: In weight 1/2, a Rademacher sum defines a function F in $\mathbb{H}$ and $\mathbb{H}^-$, satisfying

mock

$$F(\tau) = \begin{cases} f(\tau) & \text{when } \tau \in \mathbb{H} \\ \tilde{g}(-\tau) & \text{when } \tau \in \mathbb{H}^-. \end{cases}$$

Eichler int. of shad(f) (false θ)



[Cheng–Duncan 13, Rhoades 18, CCFGH]

Recall:

$$\widehat{Z}_b(M_3; q^{-1}) \sim \widehat{Z}_b(-M_3; q)$$

What's this? false–mock pair

How does that compare with expectations from CS?

$$Z_\alpha^{\text{pert}}(M_3; k) = \sum_n a_n \left(\frac{1}{k}\right)^n$$

$$\Leftrightarrow Z_\alpha^{\text{pert}}(-M_3; k) = \sum_n a_n \left(-\frac{1}{k}\right)^n$$

$q \leftrightarrow q^{-1}$

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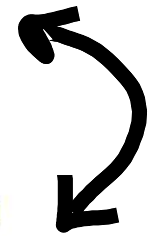


$q \leftrightarrow q^{-1}$

$$\Leftrightarrow Z_\alpha^{\mathrm{pert}}(-M_3; k) = \sum_n a_n \left(-\frac{1}{k}\right)^n$$

Up to possibly a "modular correction" G_0,
the asymp. series are just right!

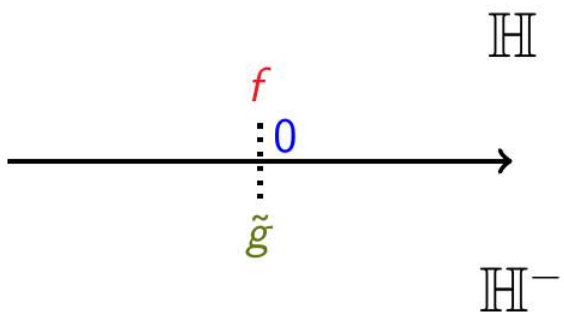
$$(f - G_0)(it) \sim \sum_{n\geq 0} a_n(-t)^n \text{ and } \tilde{g}(it) \sim \sum_{n\geq 0} a_n t^n$$

[Choi–Lim–Rhoades 16, CCFGH]

III. False, Mock, Quantum: $q \leftrightarrow q^{-1}$/mock/**quantum asymptotics**

Another way to say this :
f and $\tilde{g}$ defines the same *quantum modular form*.

eg. q-hypergeometric



$$\psi(q) := \frac{q^{\frac{1}{24}}}{2}\left(1 - \sum_{n\geq 1} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(1+q)(1+q^2)\dots(1+q^n)}\right)$$

converges when $|q| < 1$ or $|q| > 1$

$$= \Psi_1^{6+2}(\tau) = q^{\frac{1}{24}}(1 - q + q^2 + \dots)$$

$$\psi(q^{-1}) = \frac{q^{\frac{1}{24}}}{2} f_R(\tau)$$

$$\Psi_1^{(6+2)}(it) = \textstyle\sum_n a_n t^n; \;\; f_R(e^{-2\pi t}) = \sum_n a_n(-t)^n$$

$$\Rightarrow \widehat{Z}_1\left(-M(-2; \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}); q\right) \sim f_R(q)$$

Conclusions

⋆ **Resurgence & $\mathbf{SL}_2(\mathbb{Z})$ rep.** help making top. predictions on flat connections.

⋆ **False/mock pairs** help predicting $\widehat{Z}_b$ when other means aren't available.

⋆ **Quantum modular** structure guarantees the Z_{CS} interpretation.

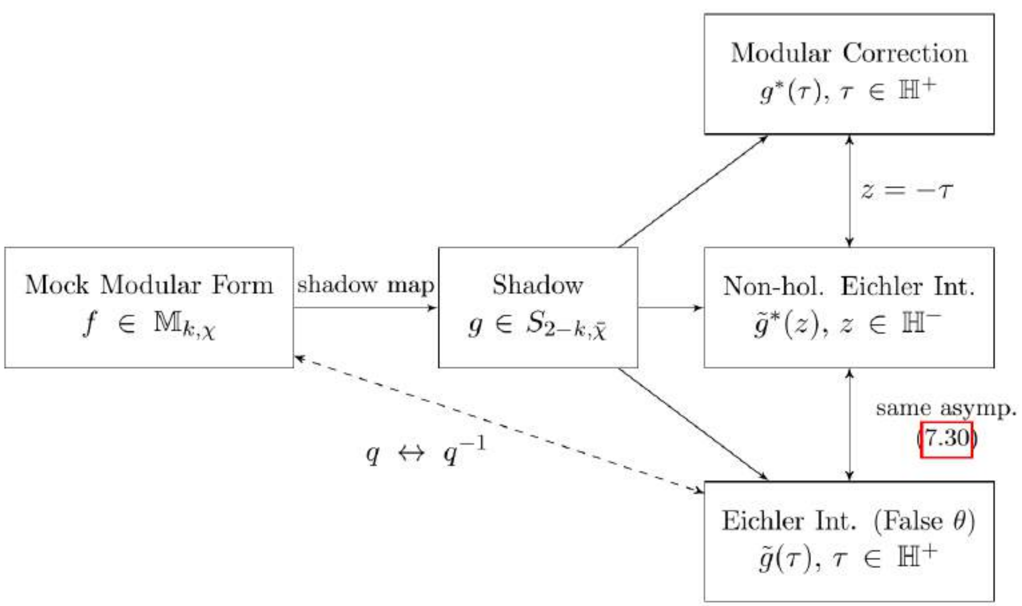
⋆ In this talk we focus on Seifert w 3 singular fibers, but similar treatment applies to other Seifert manifolds. Hyperbolic 3-manifolds aren't expected to have blocks related to false/mock but should still be quantum.

Open Questions

⋆ How do we understand the "symmetry" given by K from 3-manifolds or physics?

⋆ The mock forms are not uniquely determined by the false. Sometimes there are natural choices but sometimes we do not know. Need more topological/physical input.

Quantum Modular Forms



Open Questions

⋆ How do we understand the "symmetry" given by K from 3-manifolds or physics?

⋆ The mock forms are not uniquely determined by the false. Sometimes there are natural choices but sometimes we do not know. Need more topological/physical input.

⋆ The "modular substraction" is mysterious from the physical point of view. Can we specify this ?