Knots-quivers correspondence, lattice paths, and rational knots

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Ingredient 1: Knots

Colored HOMFLY–PT polynomials:

Symmetric (S^r)-colored HOMFLY–PT polynomials are 2-variable invariants of knots:

$$P_r(K)(a,q).$$

For $a = q^N$ they are $(sl(N), S^r)$ quantum polynomial invariants:

$$P(a = q^N, q) = P^{sl(N),S^r}(q).$$

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Already interesting is the "bottom row": the coefficient of the lowest nonzero power of a appearing in $P_r(a,q)$

$$P_r^-(q) = \lim_{a\to 0} a^\sharp P_r(a,q)$$

LMOV conjecture

Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot K is:

$$P(x,a,q) := \sum_{r\geq 0} P_r(a,q)x^r = \exp\left(\sum_{n,r\geq 1} \frac{1}{n} f_r(a^n,q^n)x^{rn}\right),$$

$$f_r(a,q) = \sum_{i,j} \frac{N_{r,i,j}a^i q^j}{q-q^{-1}}.$$

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$N_{r,i,j}$ are BPS numbers. They represent (super)-dimensions of certain homological groups. Physicaly, they "count" particles of certain type (therefore are integers).

Ingredient 2: Quivers (and their representations)

Quivers are oriented graphs, possibly with loops and multiple edges.

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Let $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a dimension vector.
We are interested in moduli space of representations of Q with the dimension vector $\mathbf{d}$:

$$M_{\mathbf{d}} = \left\{ R(\alpha) : \mathbb{C}^{d_i} \to \mathbb{C}^{d_j} | \mathit{for\ all}\ \alpha : i \to j \in Q_1 \right\} // G,$$

where $G = \prod_i GL(d_i, \mathbb{C})$.

Quivers and motivic generating functions

C is a matrix of a quiver with m vertices.

$$P_C(x_1,\ldots,x_m) := \sum_{d_1,\ldots,d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}} x_1^{d_1}\cdots x_m^{d_m}.$$

q-Pochhamer symbol $\quad (q^2;q^2)_n := \prod_{i=1}^n (1-q^{2i}).$

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Motivic (quantum) Donaldson-Thomas invariants $\Omega_{d_1,\ldots,d_m;j}$ of a symmetric quiver Q:

$$P_C = \prod_{(d_1,\ldots,d_m)\neq 0} \prod_{j\in\mathbb{Z}} \prod_{k\geq 0} \Big(1-\big(x_1^{d_1}\cdots x_m^{d_m}\big) q^{j+2k+1}\Big)^{(-1)^{j+1}\Omega_{d_1,\ldots,d_m;j}}.$$

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Theorem (Kontsevich-Soibelman, Efimov)

$\Omega_{d_1,\ldots,d_m;j}$ *are nonnegative integers.*

Knots–quivers correspondence

[P. Kucharski, M. Reineke, P. Sulkowski, M.S., Phys. Rev. D 2017]

New relationship between HOMFLY–PT / BPS invariants of knots and motivic Donaldson-Thomas invariants for quivers

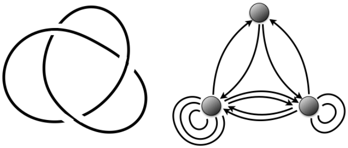

Figure: Trefoil knot and the corresponding quiver.

The generating series of HOMFLY-PT invariants of a knot matches the motivic generating series of a quiver, after setting $x_i \to x$.

Details of the correspondence

Knots	Quivers
Generators of HOMFLY homology	Number of vertices
Homological degrees, framing	Number of loops
Colored HOMFLY-PT	Motivic generating series
LMOV invariants	Motivic DT-invariants
Classical LMOV invariants	Numerical DT-invariants
Algebra of BPS states	Cohom. Hall Algebra

BPS/LMOV invariants of knots are refined through motivic DT invariants of a corresponding quiver, and so

Theorem

For all knots for which there exists a corresponding quiver, the LMOV conjecture holds.

Application 2 – Lattice paths counting

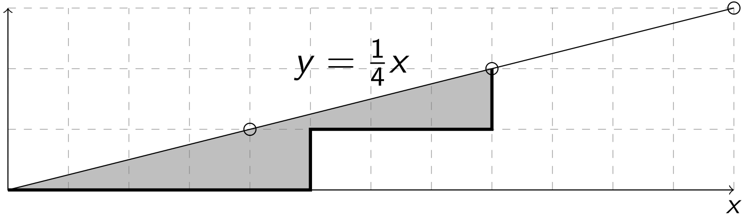

Figure: A lattice path under the line $y = \frac{1}{4}x$, and a shaded area between the path and the line.

$$y_P(x) = \sum_{k=0}^{\infty} \sum_{\pi \in k\text{-paths}} x^k = \sum_{k=0}^{\infty} c_k(1)x^k,$$

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in k\text{-paths}} q^{area(\pi)} x^k = \sum_{k=0}^{\infty} c_k(q)x^k.$$

Counting lattice paths – equivalent formulation

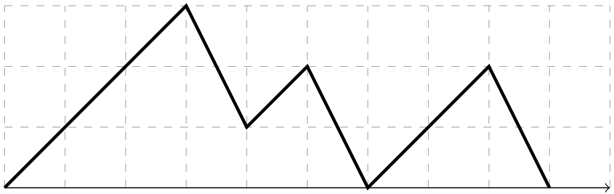

Figure: Counting of paths under the line $y = \frac{1}{2}x$ is equivalent to counting paths in the upper half plane, made of elem. steps $(1,1)$ and $(1,-2)$.

Counting (rational) lattice paths

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Observe quotient:

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Finally, take $q \to 1$ limit ("classical" limit):

$$y(x) = \lim_{q\to 1} \frac{P(K)(q;q^2x)}{P(K)(q;x)} = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

Counting (rational) lattice paths

[M. Panfil, P. Sulkowski, M.S., 2018]

Proposition

Let r and s be mutually prime.
Let $K = T_{r,s}^{f=-rs}$ be the (rs)-framed (r, s)-torus knot.
Then the corresponding coefficients a_n are equal to the number of directed lattice path from $(0, 0)$ to (sn, rn) under the line $y = (r/s)x$.

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Knots and quivers – results

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For example, for $2/3$ slope, after computing the relevant invariants (for the bottom row) of the $T_{2,3}$ knot, and applying all the machinery, we get that the matrix of a quiver corresponding to the knot is:

$$\begin{bmatrix} 7 & 5 \\ 5 & 5 \end{bmatrix}.$$

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$$\begin{aligned} a_n^{(2/3)} &= \sum_{i+j=n} \frac{1}{7i+5j+1} \binom{7i+5j+1}{i} \binom{5i+5j+1}{j} \\ &= \sum_{i=0}^{n} \frac{1}{5n+i+1} \binom{5n+2i}{i} \binom{5n+1}{n-i}. \end{aligned}$$

(rediscovered Duchon formula)

Paths under the line with slope 2/3

$$a_n^{(2/3)} = \sum_{i+j=n} \frac{1}{7i+5j+1} \binom{7i+5j+1}{i} \binom{5i+5j+1}{j}.$$

$$a_n^{(2/3)} : \quad 1, 2, 23, 377, \ldots$$

j \ i	0	1	2	3
3	35	1330	37700	$\cdots$
2	5	120	2520	50375
1	1	11	152	2275
0	1	1	7	70

New results: For 2/5 slope, i.e. $T_{2,5}$ the corresponding quiver matrix is:

$$\begin{bmatrix} 11 & 9 & 7 \\ 9 & 9 & 7 \\ 7 & 7 & 7 \end{bmatrix}.$$

$$a_n^{(2/5)} = \sum_{i+j+k=n} \frac{1}{11i+9j+7k+1} \binom{11i+9j+7k+1}{i} \binom{9i+9j+7k+1}{j} \binom{7i+7j+7k+1}{k}$$

Consequence 1 – binomial identities

All this also rediscovers some binomial identities, like e.g.

$$\binom{5n}{2n} = \sum_{i=0}^{n} \frac{5n}{5n+2i} \binom{5n+2i}{i} \binom{5n}{n-i}$$

Comes from counting of paths under line with slope $2/3$.

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Comes from counting of paths under line with slope $2/3$.

One can obtain such identities precisely for the quivers that correspond to (torus) knots.

Counting of weighted lattice paths

Proposition

The generating function $y_{qP}(x)$ of lattice paths under the line of the slope r/s, weighted by the area between this line and a given path, is equal to

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \, k\text{-}paths} q^{area(\pi)} x^k = \frac{P_C(q^2 x_1, \ldots, q^2 x_m)}{P_C(x_1, \ldots, x_m)} \Big|_{x_i = xq^{-1}}.$$

For the line of the slope r/s, the quiver in question is defined by the matrix C that encodes extremal invariants of left-handed (r, s) torus knot in framing rs.

Example: line of slope 2/3

$$
\begin{aligned}
y_{qP}(x) = {} & 1 + (q^4 + q^6)x + (q^8 + 3q^{10} + 4q^{12} + 4q^{14} + 4q^{16} + 3q^{18} + \\
& + 2q^{20} + q^{22} + q^{24})x^2 + \\
& + (q^{12} + 5q^{14} + 12q^{16} + 20q^{18} + 28q^{20} + 34q^{22} + 37q^{24} + \\
& + 37q^{26} + 36q^{28} + 33q^{30} + 29q^{32} + 25q^{34} + 21q^{36} + \\
& + 17q^{38} + 13q^{40} + 10q^{42} + 7q^{44} + 5q^{46} + \\
& + 3q^{48} + 2q^{50} + q^{52} + q^{54})x^3 + \dots \\
& \xrightarrow[q \to 1]{} 1 + 2x + 23x^2 + 377x^3 + \dots
\end{aligned}
$$

Formulae for classical DT quiver invariants

$$P_C(x_1,\ldots,x_m)=\sum_{d_1,\ldots,d_m}\frac{(-q)^{\sum_{i,j=1}^m C_{i,j}d_id_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}}x_1^{d_1}\cdots x_m^{d_m}.$$

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$$y(x_1, \ldots, x_m) = \lim_{q \to 1} \frac{P_C(q^2 x_1, \ldots, q^2 x_m)}{P_C(x_1, \ldots, x_m)} = \sum_{l_1,\ldots,l_m} b_{l_1,\ldots,l_m} x_1^{l_1} \ldots x_m^{l_m}.$$

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Coefficients $b_{l_1,\dots,l_m}$ take form

$$b_{l_1,\dots,l_m}=A(l_1,\dots,l_m)\prod_{j=1}^m\frac{(-1)^{(C_{i,i}+1)l_i}}{1+\sum_{i=1}^m C_{i,j}l_i}\binom{1+\sum_{i=1}^m C_{i,j}l_i}{l_j}$$

where

$$A(l_1,\dots,l_m)=1+\sum_{k=1}^{m-1}\sum_{admissible\,\Sigma_k}\prod_{(i_u,j_u)\in\Sigma_k}C_{i_u,j_u}l_{i_u}.$$

C is an $m \times m$ matrix.

Definition

Let $k \in \{1, \ldots, m\}$. For a set of k pairs $\{(i_1, j_1), \ldots, (i_k, j_k)\}$, with $1 \leq i_u, j_u \leq m$, we say that it is *admissible*, if it satisfies the following two conditions:

(1) there are no two equal among $j_1, \ldots, j_k$

(2) there is no cycle of any length: for any l, $1 \leq l \leq k$, there is no subset of l pairs (i_{u_ℓ}, j_{u_ℓ}), $\ell = 1, \ldots, l$, such that $j_{u_\ell} = i_{u_{\ell+1}}$, $\ell = 1, \ldots, l-1$, and $j_{u_l} = i_{u_1}$.

Alternative, recursive definition

We observe the set of polynomials in m variables $p(C)(x_1,\ldots,x_m)$, whose coefficients depend are functions of the entries of matrix C.

Define actions of the symmetric group $\mathbb{S}_m$ on $m\times m$ matrices by:

$$[\sigma\circ C]_{i,j} := C_{\sigma_i,\sigma_j}, \qquad i,j=1,\ldots,m,$$

and on polynomials $p(C)(x_1,\ldots,x_m)$ by

$$\sigma\circ p(C)(x_1,\ldots,x_m) := p(\sigma\circ C)(x_{\sigma_1},\ldots,x_{\sigma_m}).$$

Alternative, recursive definition

Then, for a given $m \times m$ matrix C we define the polynomial $P_m(C)(x_1, \ldots, x_m)$ by the following:

- $\sigma \circ P_m(C)(x_1, \ldots, x_m) = P_m(C)(x_1, \ldots, x_m), \quad \forall \sigma \in \mathbb{S}_m$
- $P_1(C)(x) = 1,$
- $P_m(C)(x_1, \ldots, x_{m-1}, 0) = P_{m-1}(C')(x_1, \ldots, x_{m-1})\left(1 + \sum_{i=1}^{m-1} C_{i,m} x_i\right).$

where C' denotes the submatrix of C formed by its first $m - 1$ rows and columns.

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Then:

$$A(l_1, \ldots, l_m) = P_m(C)(l_1, \ldots, l_m).$$

Paths under the line with slope 3/4

$$C^{(3,4)} = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 \\ 7 & 9 & 8 & 9 & 9 \\ 7 & 8 & 9 & 9 & 10 \\ 7 & 9 & 9 & 11 & 11 \\ 7 & 9 & 10 & 11 & 13 \end{bmatrix}$$

$$\sharp paths = \sum_{l_1+\cdots+l_5=n} A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)\times$$
$$\times \frac{1}{7l_1+7l_2+7l_3+7l_4+7l_5+1}\binom{7l_1+7l_2+7l_3+7l_4+7l_5+1}{l_1}\times$$
$$\times \frac{1}{7l_1+9l_2+8l_3+9l_4+9l_5+1}\binom{7l_1+9l_2+8l_3+9l_4+9l_5+1}{l_2}\times$$
$$\times \frac{1}{7l_1+8l_2+9l_3+9l_4+10l_5+1}\binom{7l_1+8l_2+9l_3+9l_4+10l_5+1}{l_3}\times$$
$$\times \frac{1}{7l_1+9l_2+9l_3+11l_4+11l_5+1}\binom{7l_1+9l_2+9l_3+11l_4+11l_5+1}{l_4}\times$$
$$\times \frac{1}{7l_1+9l_2+10l_3+11l_4+13l_5+1}\binom{7l_1+9l_2+10l_3+11l_4+13l_5+1}{l_5}.$$

$$
\begin{aligned}
A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)= {} & 1+28\, l_1+294\, l_1^2+1372\, l_1^3+2401\, l_1^4+33\, l_2+693\, l_1 l_2+4851\, l_1^2 l_2+11319\, l_1^3 l_2+407\, l_2^2+5698\, l_1 l_2^2+\\
& +19943\, l_1^2 l_2^2+2223\, l_2^3+15561\, l_1 l_2^3+4536\, l_2^4+34\, l_3+714\, l_1 l_3+4998\, l_1^2 l_3+11662\, l_1^3 l_3+838\, l_2 l_3+11732 l_1 l_2 l_3+\\
& +41062 l_1^2 l_2 l_3+6860 l_2^2 l_3+48020 l_1 l_2^2 l_3+18648 l_2^3 l_3+431 l_3^2+6034 l_1 l_3^2+21119 l_1^2 l_3^2+7051 l_2 l_3^2+49357 l_1 l_2 l_3^2+\\
& +28728 l_2^2 l_3^2+2414 l_3^3+16898 l_1 l_3^3+19656 l_2 l_3^3+5040 l_3^4+36 l_4+756 l_1 l_4+5292 l_1^2 l_4+12348 l_1^3 l_4+887 l_2 l_4+\\
& +12418 l_1 l_2 l_4+43463 l_1^2 l_2 l_4+7258 l_2^2 l_4+50806 l_1 l_2^2 l_4+19719 l_2^3 l_4+21294 l_3^3 l_4+482 l_4^2+6748 l_1 l_4^2+23618 l_1^2 l_4^2+\\
& +912 l_3 l_4+12768 l_1 l_3 l_4+44688 l_1^2 l_3 l_4+14914 l_2 l_3 l_4+104398 l_1 l_2 l_3 l_4+60732 l_2^2 l_3 l_4+7656 l_3^2 l_4+53592 l_1 l_3^2 l_4+62307 l_2 l_3^2 l_4+\\
& +7879 l_2 l_4^2+55153 l_1 l_2 l_4^2+32067 l_2^2 l_4^2+8086 l_3 l_4^2+56602 l_1 l_3 l_4^2+23688 l_3 l_4^3+6237 l_4^4+65772 l_2 l_3 l_4^2+37 l_5+777 l_1 l_5+\\
& +33705 l_3^2 l_4^2+2844 l_4^3+19908 l_1 l_4^3+23121 l_2 l_4^3+5439 l_1^2 l_5+12691 l_1^3 l_5+912 l_2 l_5+12768 l_1 l_2 l_5+44688 l_1^2 l_2 l_5+\\
& +7465 l_2^2 l_5+52255 l_1 l_2^2 l_5+20286 l_2^3 l_5+69524 l_2 l_3 l_5^2+35630 l_3^2 l_5^2+9010 l_4 l_5^2+63070 l_1 l_4 l_5^2+39501 l_4^2 l_5^2+25074 l_2 l_5^3+\\
& +938 l_3 l_5+13132 l_1 l_3 l_5+45962 l_1^2 l_3 l_5+15342 l_2 l_3 l_5+107394 l_1 l_2 l_3 l_5+62482 l_2^2 l_3 l_5+7877 l_3^2 l_5+3083 l_5^3+21581 l_1 l_5^3+\\
& +55139 l_1 l_3^2 l_5+64106 l_2 l_3^2 l_5+21910 l_3^3 l_5+991 l_4 l_5+13874 l_1 l_4 l_5+48559 l_1^2 l_4 l_5+16204 l_2 l_4 l_5+73269 l_2 l_4 l_5^2+75068 l_3 l_4 l_5^2+\\
& +113428 l_1 l_2 l_4 l_5+65961 l_2^2 l_4 l_5+16632 l_3 l_4 l_5+116424 l_1 l_3 l_4 l_5+135296 l_2 l_3 l_4 l_5+69335 l_3^2 l_4 l_5+25690 l_3 l_5^3+\\
& +8771 l_4^2 l_5+61397 l_1 l_4^2 l_5+71316 l_2 l_4^2 l_5+73066 l_3 l_4^2 l_5+25641 l_4^3 l_5+509 l_5^2+7126 l_1 l_5^2+27027 l_4 l_5^3+\\
& +24941 l_1^2 l_5^2+8325 l_2 l_5^2+58275 l_1 l_2 l_5^2+33894 l_2^2 l_5^2+8546 l_3 l_5^2+59822 l_1 l_3 l_5^2+6930 l_5^4 .
\end{aligned}
$$

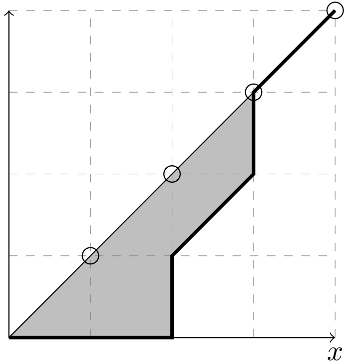

Figure: An example of a Schröder path of length 6.

Schröder paths and full colored HOMFLY-PT

Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing $f = 1$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This corresponds to counting paths under the diagonal line $y = x$.

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Then from the quiver generating function of C we get

$$y(x, a, q) = 1 + (q + a)x + \left(q^2 + q^4 + (2q + q^3)a + a^2\right)x^2 + \dots$$

with the height of a path measured by the power of x and the number of diagonal steps measured by the power of a.

Schröder paths and full colored HOMFLY-PT

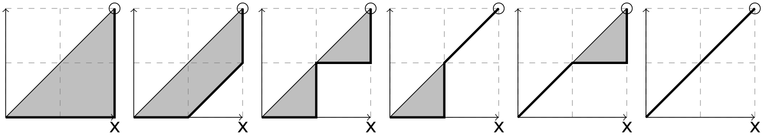

Figure: All 6 Schröder paths of height 2 represented by the quadratic term $q^2 + q^4 + (2q + q^3)a + a^2$ of the generating function.

Consequence 2 – Some divisibilities (integrality)

If p is prime, then:

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Corresponds to the fact that DT invariants are non-negative integers (in this case of the quiver of the framed unknot — one vertex, m-loop quiver)

Consequence 2 – Integrality of DT invariants

Let C be a 2-vertex quiver:

$$C = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

with $\alpha, \beta, \gamma \in \mathbb{N}$.

Then for every $r, s \in \mathbb{N}$ we have

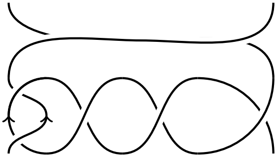
$$\Omega_{r,s} = \frac{\beta}{(\alpha r + \beta s)(\beta r + \gamma s)} \sum_{d \mid \gcd(r,s)} (-1)^{(\alpha+1)r/d+(\gamma+1)s/d} \mu(d) \times$$
$$\times \binom{\alpha r/d + \beta s/d}{r/d} \binom{\beta r/d + \gamma s/d}{s/d} \in \mathbb{N}$$

Rational knots

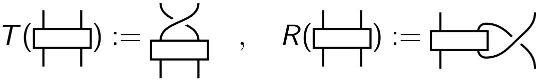
(joint with P. Wedrich)

$$p/q = [a_1, \ldots, a_r] = a_r + \cfrac{1}{a_{r-1} + \frac{1}{a_{r-2}+\ldots}}$$

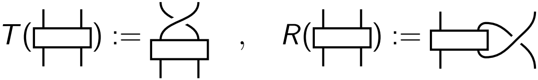
Rational tangle encoded by $[2, 3, 1]$

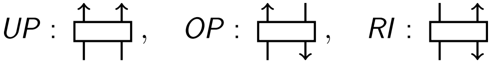


Rational knots

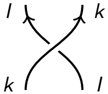


Rational knots

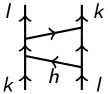


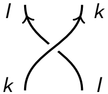


Skein theory

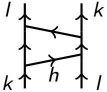


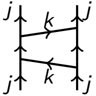
$\overset{k \geq l}{\equiv} \sum_{h=0}^{l} (-q)^{h-l}$

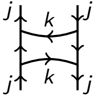


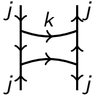


$\overset{k \leq l}{\equiv} \sum_{h=0}^{k} (-q)^{h-k}$



$UP[j,k] =$ 

, $OP[j,k] =$ 

, $RI[j,k] =$ 

1. $TUP[j,k] = \sum_{h=k}^{j} (-q)^{h-j} q^{k^2} \begin{bmatrix} h \\ k \end{bmatrix}_+ UP[j,h]$
2. $RUP[j,k] = \sum_{h=0}^{k} (-q)^{h-j} a^{h-j} q^{-2kh+k^2+j^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ OP[j,h]$
3. $TOP[j,k] = \sum_{h=k}^{j} (-q)^{h} a^{k} q^{k^2-2jk} \begin{bmatrix} h \\ k \end{bmatrix}_+ RI[j,h]$
4. $ROP[j,k] = \sum_{h=0}^{k} (-q)^{h-j} a^{k-j} q^{2h(j-k)+(k-j)^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ UP[j,h]$
5. $TRI[j,k] = \sum_{h=k}^{j} (-q)^{h} a^{h} q^{k^2-2jh} \begin{bmatrix} h \\ k \end{bmatrix}_+ OP[j,h]$
6. $RRI[j,k] = \sum_{h=0}^{k} (-q)^{h} q^{h(2j-2k)+k^2-j^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ RI[j,h]$

Rational knots: Knots-quivers corresponds works!

Theorem

Let K be a rational knot and let Q_K be the corresponding quiver. Then, the vertices of Q_K are in bijection with generators of the reduced HOMFLY-PT homology of K, such that the (a, q, t)-trigrading of the i^{th} generator is given by $(a_i, -Q_{i,i} - q_i, -Q_{i,i})$ where $Q_{i,i}$ denotes the number of loops at the i^{th} vertex of Q_K.

Open questions – work in progress

- Find quivers for other larger classes of knots
- How to find a quiver for a given knot directly (geometrically, topologically...)? Other, better definition?
- The (non)uniqueness of a quiver.
- More proofs....
- Closed formulas for q-version of paths.
- Further integralities and combinatorial identities, Rogers-Ramanujan identities,...
- Links and quivers...
- Extend to all representations, not necessarily symmetric.
- What is so special for quivers that correspond to knots? Combinatorial identities for binomial coefficients, and extended integrality/divisibility hold precisely for them.

Thank you for your attention !