Khovanov homotopy type for periodic links

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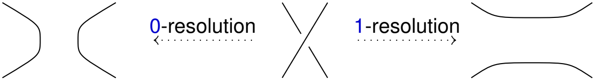

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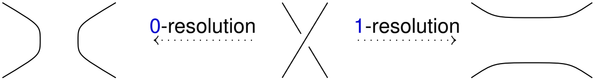

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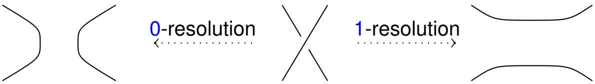

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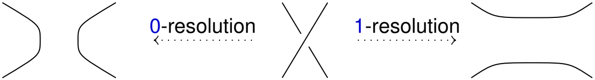
- The differentials are elementary cobordisms.



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- We have grading obtained from q.

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- Or one circle is split.

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- The differential is defined with these maps (up to sign).

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- Detects the unknot (Kronheimer, Mrowka 2007);
- Allows to compute the smooth four-genus of torus knots (Rasmussen, 2003) via s-invariants.

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- How can it be constructed?

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- No! $\mathbb{C}P^2 \sqcup \mathbb{C}P^2$ and $S^4 \sqcup S^2 \times S^2$ admit Morse functions with 2 minima, 2 maxima and 2 critical points of index 2. The Morse complex is trivial for dimensional reasons.

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- Need to incorporate moduli spaces of trajectories.

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Set $M = [0, 1]^n$ and $f(x_1, \dots, x_n) = \sum f(x_i)$, where $f(x) = -x^3 + 3x^2$.

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- there are various compatibility relations of the composition map.

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Does any Morse flow category determine the underlying manifold?

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- Then perform Cohen-Jones-Segal construction.
- Different, more specific: define an appropriate functor from $\mathcal{C}$ to a cube category (cover) and use the embedding of $\mathsf{Cube}(n)$.

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- Our aim is to define $\mathcal{M}(x, y)$ for all x, y such that $x \prec y$.

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- Based on this functor one can define a framing and perform a construction of $\mathcal{X}_D$.

Periodic knots

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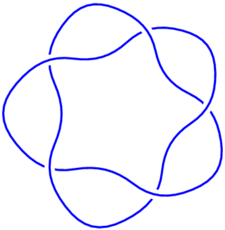

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- Does not depend on the choice of the diagram.

Equivariant Khovanov homology

- For a periodic diagram D, the group $\mathbb{Z}_p$ permutes the cube of resolution.
- Take a ring R. Then $\mathrm{CKh}(D;R)$ has a structure of $\Lambda = R[\mathbb{Z}_p]$–module.

Definition (Politarczyk)

For any Λ–module M define the equivariant Khovanov homology as

$$\mathrm{EKh}(K;M) = \mathrm{Ext}_\Lambda(M, \mathrm{CKh}(D;R)).$$

- Does not depend on the choice of the diagram.
- Most important example: $M = \Lambda$.

Equivariant Khovanov. Properties.

- We can define $\mathrm{EKh}_d(L) = \mathrm{EKh}(L; \mathbb{Z}[\xi_d])$ for any $d|p$. This is the third gradation, coming from representations of $\mathbb{Z}_p$.

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- On the other hand we have a Schur decomposition of $\mathrm{Hom}_\Lambda(\Lambda; CKh(D))$.

Equivariant flow category

Recall the definition:

Definition

We say that $\mathcal{C}$ is a *flow category* if

- $\operatorname{Ob}\mathcal{C}$ form a finite set;
- there is a grading function $gr\colon \operatorname{Ob}\mathcal{C} \to \mathbb{Z}$;
- if $x, y \in \operatorname{Ob}\mathcal{C}$ and $y \neq x$, then $\mathcal{M}(x, y)$ is a compact $gr(y) - gr(x) - 1$-dimensional manifold with corners, $\mathcal{M}(x, x) = \{pt\}$;
- if $x, y, z \in \operatorname{Ob}\mathcal{C}$ and $gr(x) < gr(z) < gr(y)$, there is a composition map $\mathcal{M}(x, z) \times \mathcal{M}(z, y) \to \partial\mathcal{M}(x, y)$, the boundary of $\mathcal{M}(x, y)$ is all covered by such products;
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The key idea: replace the grading function to $gr\colon \operatorname{Ob}\mathcal{C} \to RO(G)$.

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- In our setting we define consistently the equivariant grading.
- The functor $\mathfrak{f}$ commutes with the group action.

Main result

Theorem (—,Politarczyk, Silvero)

Everything works.

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If L is an m-periodic link, then spaces $\mathcal{X}_L^q$ are well-defined up to stable equivariant homotopy equivalence.

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If L is an m-periodic link, then spaces $\mathcal{X}_L^q$ are well-defined up to stable equivariant homotopy equivalence.

- The proof is much more involved;
- Invariance under Reidemeister moves uses the fact that the cube category Cube admits a group action and for any $H \subset \mathbb{Z}_m$ the fixed point category Cube^H is again the cube category.

Theorem (—,Politarczyk, Silvero)

Let L be an m-periodic link and suppose $\mathbb{F}$ is a field. For any R-torsion-free $R[\mathbb{Z}_m]$-module M we have an isomorphism of $R[\mathbb{Z}_m]$-modules:

$$\mathrm{EKh}^{i,q}(L;M) \cong \widetilde{H}^{*}_{G}i(\mathcal{X}_L^q, \mathrm{Hom}_R(M,R)).$$

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- Steenrod squares commute with group actions;
- Refinement of Borodzik–Politarczyk periodicity criterion;
- Potential insight into Khovanov homology of periodic links, like torus links.

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Which deals with something entirely different.