String Theory And Homological Invariants for 3-Manifolds

Cumrun Vafa
Harvard University

University of Warsaw
September 24, 2018

Based on:

S.Gukov, P.Putrov and C.Vafa,
``Fivebranes and 3-manifold homology,'' arXiv:
1602.05302 [hep-th].

S.Gukov, D.Pei, P.Putrov and C.Vafa,
``BPS spectra and 3-manifold invariants,''
arXiv:1701.06567 [hep-th].

S. Gukov, D. Pei, P. Putrov and C. Vafa, to appear.

Plan for this talk:

1-Review connection between knot homology and physics (old work)

2-Physical setup for homological invariants for 3-manifolds (new work)

3-Examples

4-Extensions

Physical Setup for Knot Homology:

We start from 6d object (M5 branes) [(2,0) ADE type]

$R^3 \times \quad S^3$

Physical Setup for Knot Homology:

We start from 6d object (M5 branes) [(2,0) ADE type]

$$\mathbb{R}^3 \times \left[S^3 \cap T^* S^3 \right]$$

Physical Setup for Knot Homology:

We start from 6d object (M5 branes) [(2,0) ADE type]

[Ooguri,V]

The resulting 3d theory has an associated Hilbert space; CS Knot invariants can be reformulated in terms of character of one of the gradations (`spin') of this Hilbert space. [Ooguri,V]

More generally there is an extra gradation (`R-symmetry') in this Hilbert space that can be used to define Homological invariants (making contact with Khovanov-Rozhansky homology). [Gukov,Schwartz,V] And in certain cases to a ``refined CS theory.'' [Aganagic,Shakirov]
Moreoever, using string dualities the states of this Hilbert space can be mapped to solutions of certain gauge theory equations. [Witten]

$$\mathcal{H}^{K}_{r_1, r_2} \qquad r_1, r_2 \in \mathbb{Z}$$

HOMFLY:

$$Z_K(q) = \mathrm{Tr}\,(-)^{r_2}\, q^{r_1}$$

Poincare $\mathrm{Tr}(-)^{r_2} q^{r_1} t^{r_2} = P_K(q, t)$

For special knots (such as torus knots) we end up with extra symmetry, leading to additional gradation. In such cases, often we can recover the Poincare polynomial as an index:

extra

$$\mathcal{H}_{r_1, r_2, r_3}$$

$$\mathrm{Tr}\, (-)^{r_2} q^{r_1} t^{r_3} = P_K(q,t)$$

$$\mathcal{H}\{r_3 \neq r_2\} = \phi$$

The basic idea to define homological invariants for three manifolds closely follows the above idea:

Start with 6d theory (2,0) ADE theory and compactify on the 3-manifold X. The resulting 3d theory (6-3=3) will have a Hilbert space which can be used to define homological invariants associated to X.
We then use this to recast WRT invariants in this way.
Moreover, we can use the extra gradation on the Hilbert space to find a refined extension of WRT.

In order to better understand this, we need to explain the relevant physics background; But before that let me summarize the statements:

For simplicity of presentation let me assume rational homology 3-spheres; i.e.

$$b_1(X) = 0$$

Consider

$$a: \quad H_1(X, \mathbb{Z}) \longrightarrow C^{ADE} / W_{ADE}$$

Where C denotes the Cartan torus of ADE and W denotes the Weyl group. We can think of it as abelian flat ADE connections on X.

For each three manifold X and each class of abelian flat connection `a' we get homological invariants:

① $\mathcal{H}^X_{a;\, r_1, r_2}$

$r_i \in \mathbb{Z}$

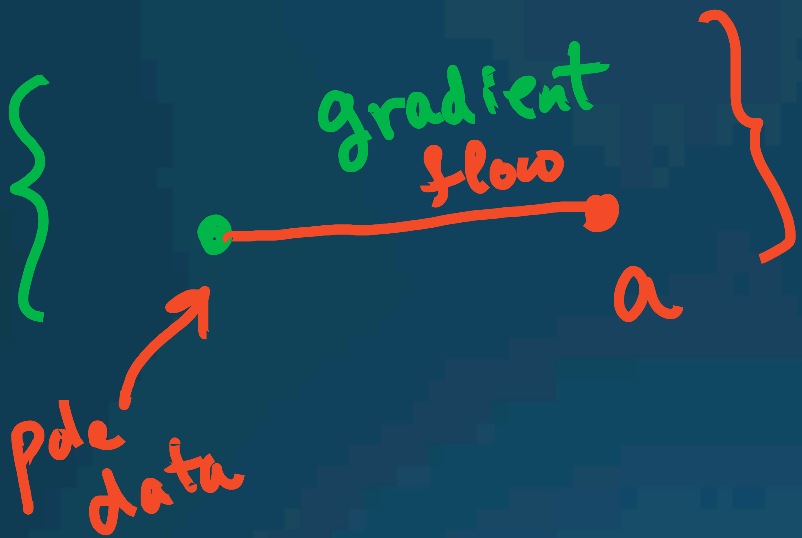
$$P_a(q,t) = \mathrm{Tr}_{\mathcal{H}_a} (-)^{r_2} q^{r_1} t^{r_2}$$

Mathematically more precise definition:
Consider complexified gauge connections on X;
There is a natural gradient flow on this space using the Chern-Simons action on X: W=CS(A).
For each flat connection `a' there is a Lagrangian subspace of A (which can be constructed using Lefshetz thimble) [Witten]. Moreover we can consider a canonical boundary condition in which as x-->o

$$A_\mu \longrightarrow \sum_{a=1}^{3} \frac{e_\mu^a\, t_a}{x} \qquad g_{\mu\nu} = e_\mu^a e_\nu^a$$

t_a: $\mathrm{Lie}(SU(2)) \longrightarrow \mathrm{Lie}(G)$
(principal embedding)

$$\mathcal{H}_{a;r_1,r_2} \equiv$$



$$r_1 = \int_{I\times X} F\wedge F$$

r_2 = R-charge

$$Z_a(q) \equiv P_a(q, t=1)$$

roughly:

Partition function of CS near flat `a' connection.

$$Z^{WRT}(X) = \sum_{a,b} \frac{e^{i\pi k \tilde{\ell}(a,a) + 2i\pi k \tilde{\ell}(a,b)}}{|G_a|\, |H_1(X,\mathbb{Z})|^{\frac{r}{2}}} \cdot Z_b(q)$$

$$q = e^{\frac{2\pi i}{k}} \qquad G_a = \mathrm{Stab}(a)$$

$\tilde{\ell}$ comes from ℓ: $H_1 \times H_1 \to \mathbb{Q}/\mathbb{Z}$ & Killing form

Similarly when refined Chern-Simons theory is defined we have

$$Z_X^{\text{ref. CS}}(q,t) = \sum_{a,b} M_{a,b} Z_b(q,t)$$

These statements coming from physics considerations can be viewed as motivations for further exploration of the mathematical definition of it I just gave.

I now describe what the physics motivation for this comes from and give some examples.

As we will see, the story is very much in parallel to how the homological invariants for links are realized in the M-theory setup.

Consider putting the 6d (2,0) theory on:

$$X \times R^3$$

$$\Downarrow \quad g_X \to 0$$

$$R^3$$

SCFT

In many cases the resulting theory can be effectively described in terms of low energy limit of quiver gauge theories starting with the work of [Dimofte,Gaiotto,Gukov].

The theory has a superconformal symmetry:

$$Osp(4|2)$$

The bosonic part of this symmetry is

$$SO(3,2)xSO(2)$$

This has three independent Cartans giving three gradations for the Hilbert space.

So we get a map:

Not only does this have a Hilbert space, but much more data including state/operator correspondence leading to a product structure on the Hilbert space.

This is a lot of data, and perhaps too complicated to capture, as a way to distinguish different X's.

To describe the resulting Hilbert space we view the space-time as

$$\mathbb{R}^3 = \mathbb{R}^t \times \mathbb{R}^2$$

$$\mathbb{R}^2 = \text{space} \underset{\text{conf}}{\cong} S^2$$

To compute the partition function including gradation we can compute:

$$\mathrm{Tr} \prod_{i=1}^{3} q_i^{H_i}$$

Cartan

$SO(3,2) \times SO(2)$

$$S^1_{q_i} \ltimes S^2$$

$$\updownarrow$$

$$Z(q_i) = \mathrm{Tr} \prod_i q_i^{H_i}$$

This is in general too complicated to compute;
Moreover it may vary as we change the metric on X.
Focusing on chomolology of it resolves this issue:
Focus on subspace of states which are annihilated by
Q and its adjoint:

$$H_0 : Q|\psi\rangle = 0$$

$$Z(q) = \mathrm{Tr}(-)^F q^S$$

S : spin $SO(3,2)$

$$\{Q, Q^\dagger\} = \sum_{i=1}^{3} \alpha_i H_i$$

$$\mathcal{H}_0 \longleftrightarrow \sum \alpha_i H_i = 0$$

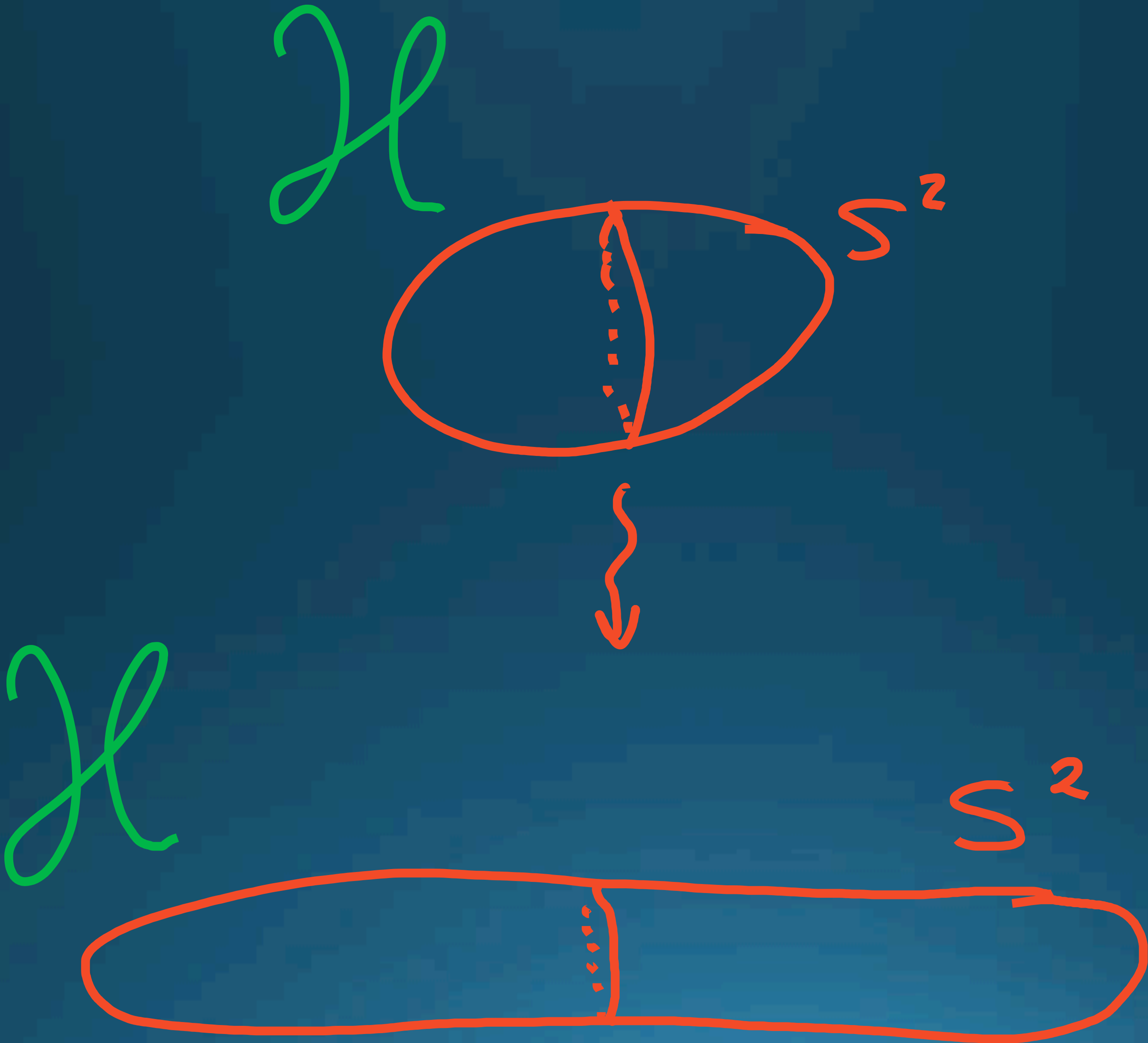
One homological invariant associated to X is given by the Q preserved subspace of the Hilbert space together with the gradation. This turns out to have a further refinement, which we will discuss later.

One of the 3 Cartan directions are frozen by being in the kernel of Q; There are two gradations from SO(3,2)xSO(2) left. One combination of SO(3,2) spin commutes with fermion number and so can be used to compute the index. The other combination which will involve the SO(2) part gives an additional gradation which in general is not an index.

This is very similar to the case of the knot invariants: The index explains the integrality of the HOMFLY polynomials and the associated Hilbert space has an extra gradation which leads to refined homological invariants.

Just as in the case of knots and links for some specific knots with extra symmetries (such as torus knots) the extra gradation sometimes correlates with an additional gradation due to symmetry again leading to an index computation:

$$\mathrm{Tr}\,(-)^F q^S t^f = Z(q,t)$$



$\mathcal{H}$
S^2
$\mathcal{H}$
S^2

$$\mathcal{H} \approx$$

$$\mathcal{H}_a \otimes \mathcal{H}_a$$

$$\mathcal{H} \sim \sum_a \mathcal{H}_a \otimes \mathcal{H}_a$$

(2,0) A on

$$X \times S^1 \ltimes_q$$

$\rightarrow$ topological strings on X

$\downarrow$

$CS(X)$

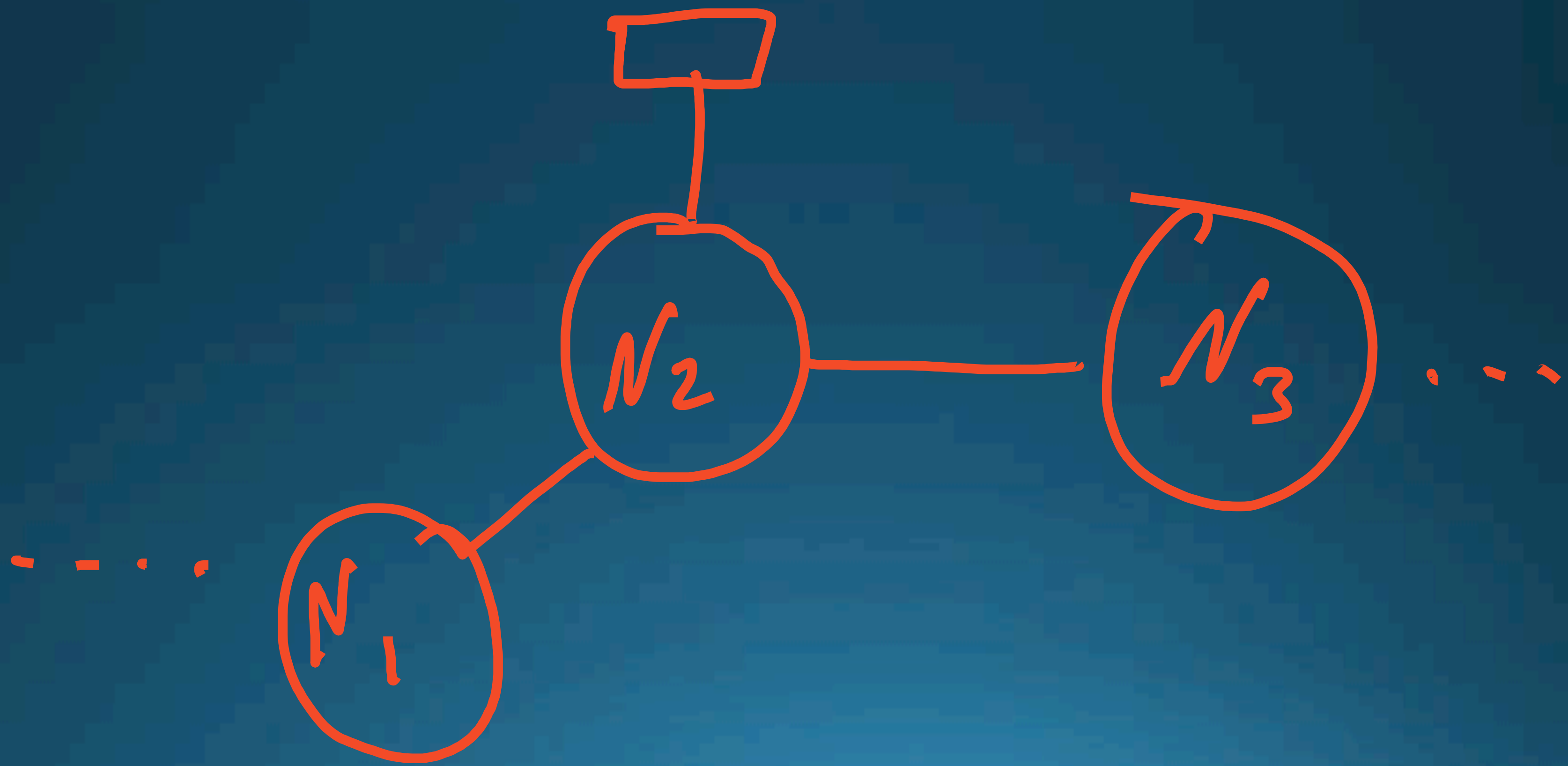
$$\mathcal{H}_{a;r_1,r_2}$$

$$Z(q,t)=\sum_a \underbrace{Z_a(q,t)}\, Z_a(q^{-1},t^{-1})$$

$$Z_a(q,t) \longleftrightarrow \mathcal{H}_{a;r_1,r_2}$$

To get a handle on this subspace we need to have an effective description of the (2,0) compactified on X.

Typically the resulting theory can be described by a quiver gauge theory:



Example: Lens space X=L(p,1)

$= S^3/\mathbb{Z}_p$

U(N)
CS

level $= p$

$$(z,q)_\infty = \prod_{n=0}^{\infty} (1 - z q^n)$$

$$Z_{N=1} = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} z^{2pm} (qt)^{-2|m|} (1 - z^{\pm 2} q^{|m|}) \times \frac{(1/t; q)_\infty (z^{\pm 2} q^{|m|}/t; q)_\infty}{(qt; q)_\infty (z^{\pm 2} q^{|m|+1} t; q)_\infty}$$

$$Z = \sum_a Z_a(q,t)\, Z_a(q^{-1},t^{-1})$$

For the three sphere we only have one block:

$$Z_0 = \prod_{i=1}^{N} \frac{(tq, q)_\infty}{(t^i q^i, q)_\infty}$$

$$Z_0(q, t \to 1) = \prod_{d=1}^{N-1} (1 - q^d)^{N-d}$$

(= WRT)

$S^3/\mathbb{Z}_p$

$$Z_a(q,t) = \frac{1}{|G_a|} \frac{1}{(tq;q)_\infty^N} \oint \frac{dz_i}{2\pi i z_i} \frac{(z_i/z_j;q)_\infty}{(z_i/z_j\, tq;q)_\infty} \theta_a(z,q)$$

Possible Extensions

-Can Include knots

-U(N|M) for N=M=1 equivalent to SW Floer homology (pinched cylinders) [Spin^c structure—flat abelian connection]

-Generalize from 6d (2,0) theories to 6d (1,0) theories, classified in terms of singularities of elliptic CY 3-folds.

These could lead to 3d theories with N=1 supersymmetry. This could lead to homological invariants with a single grading.

The fact that there is only a single grading for the homological invariant is compensated by the fact that the number of 6d (1,0) theories is far more numerous.

The index version of these invariants, which ignores the gradation, is an extension of Rozansky-Witten invariants associated to 3d sigma models to hyperKahler manifolds.

Indeed the more general thing one can do is to consider invariants for 4-manifolds:

6d(1,0)→2d (1,0) Theories

M —> TMF (topological modular forms)
(More precisely, the 6d (1,0) theories have symmetry groups—> Equivariant version of TMF)

6d (1,0) theories are essentially classified by singularities of elliptic CY 3-folds. Therefore

Sing. Elliptic CY3: M_4–>TMF (G)

Any canonical map from M_3->M_4 leads to invariants.

Conclusion

We have proposed, based on physical reasoning, a new homological invariant for three manifolds with a double grading, labeled by flat abelian connections.

We have shown how this data refines (and reduces) to WRT invariants and computed it in a number of examples.

We have argued there may be an even more extensive new homological invariants based on (1,0) 6d theories which we are currently developing.