# Chern-Simons theory and the higher genus B-model 

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Quantum fields, knots and strings, Sep 2018


## This talk

Main thread of this talk: search of large $N$ realisations of Chern-Simons invariants of $\left(M^{3}, \mathcal{K}\right)$ in the B-model.

> The overall aim is to extract all-genus results for these invariants from the resulting connection with mirror symmetry, string theory, and the topological recursion.

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\text { Message: Chern-Simons } \leftrightarrow \text { Eynard-Orantin }
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> The first part is about $U(N)$ quantum invariants of $M^{3}$ - Chern-Simons partition functions. Here, for constant positive curvature $M^{3}=S^{3} / \Gamma$,

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[AB-Borot '15; AB '17]
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\operatorname{CS}\left(S^{3}, \mathcal{K}\right) \stackrel{?}{=} \operatorname{CEO}\left(\mathcal{S}_{\mathcal{K}}\right)
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[AB, work in progress]

## Outline

(1) Review: $\left(S^{3}, \bigcirc\right)$
(2) $S^{3} \longrightarrow M^{3}$
(3) $\bigcirc \longrightarrow \mathcal{K}$

## Review: $\left(S^{3}, \bigcirc\right)$

## $U(N)$ Chern-Simons theory at large $N$

$M^{3}$ smooth, closed, oriented, $\mathcal{K} \simeq S^{1} \hookrightarrow M^{3}, k \in \mathbb{Z}^{\star}, \rho \in \operatorname{Rep}(U(N))$.

Classical action: CS[A]
Partition function: $Z_{\mathrm{CS}}^{M^{3}}(N, k)$
Wilson loops: $W_{\mathrm{CS}}^{M^{\rho}, K}(N, k, \rho)$ (Schur colouring)

$Z_{\mathrm{CS}}$ and $W_{\mathrm{CS}}$ realise gauge-theoretically the sl${ }_{N}$ RTW invariant of $M^{3}$ and $\mathcal{K}$.


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=\int_{M^{3}} \operatorname{Tr}_{\square}\left(A \wedge \mathrm{~d} A+\frac{2}{3} A^{3}\right)
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\stackrel{!}{\in} \mathbb{Z}\left[\left(q \triangleq \mathrm{e}^{\frac{2 \pi i}{k+N}}\right)^{ \pm},\left(Q \triangleq \mathrm{e}^{\frac{2 \pi i N}{k+N}}\right)^{ \pm}\right]
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$Z_{\mathrm{CS}}$ and $W_{\mathrm{CS}}$ realise gauge-theoretically the $\mathrm{sl}_{N}$ RTW invariant of $M^{3}$ and $\mathcal{K}$.

$$
M^{3} \simeq S^{3}: \quad W_{\mathrm{CS}}^{M^{3}, \mathcal{K}}(N, k, \rho) \propto H_{\mathcal{K}}^{\rho}(q, Q)
$$

## $U(N)$ Chern-Simons theory at large $N$

## Newton colouring:

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W_{\mathrm{CS}}^{M^{3}, \mathcal{K}}(N, k, \vec{d})=\left\langle\operatorname{Tr}_{\square} \operatorname{Hol}_{\mathcal{K}}\left(A^{d_{1}}\right) \ldots \operatorname{Tr}_{\square} \operatorname{Hol}_{\mathcal{K}}\left(A^{d_{h}}\right)\right\rangle^{(c)} \stackrel{!}{\in}\left[q^{ \pm}, Q^{ \pm}\right]
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## Large $N$ generating functions:



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Large $N$ generating functions:

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\omega_{g, h}^{M^{3}, \mathcal{K}}(\vec{x}, Q)=\left[(\ln q)^{2 g-2+h}\right]\left\langle\prod_{i}^{h} \operatorname{Tr}_{\square} \frac{1}{1-x_{i} \operatorname{Hol}_{\mathcal{K}}(A)}\right\rangle^{(c)} \in \mathbb{Z}\left[Q^{ \pm}\right][[\vec{x}]]
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W_{\mathrm{CS}}^{M^{3}, \mathcal{K}}(N, k, \rho) \leftrightarrow W_{\mathrm{CS}}^{M^{3}, \mathcal{K}}(N, k, \vec{d}) \leftrightarrow \omega_{g, h}^{M^{3}, \mathcal{K}}(\vec{x}, Q)
\end{gathered}
$$

Two stringy viewpoints on $\omega_{g, h}^{M^{3}, \mathcal{K}}$

Take A: counts of open holomorphic curves
"Arnold's principle": smooth invariants for $(M, S)$ via symplectic invariants of ( $T^{*} M, N_{S / M}^{*} S$ )
CS on $\left(S^{3}, \mathcal{K}\right) \longleftrightarrow$ open A-model on $\left(T^{*} S^{3}, N_{\mathcal{K}}^{*} / S^{3} \mathcal{K}\right)$

At large $(N, k)$ :
CS on $\left(S^{3}, \mathcal{K}\right) \leftarrow$ open/closed A-model on $\left(X=\operatorname{Tot}\left(O^{0^{2}}(-1)_{p 1}\right), L_{K}\right)$

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[Gopakumar-Vafa '98, Ooguri-Vafa '99]

Two stringy viewpoints on $\omega_{g, h}^{M^{3}, \mathcal{K}}$

Take $A$ : counts of open holomorphic curves
$\mathcal{K}=\bigcirc: L_{\mathcal{K}}=L_{\bigcirc}=X^{\sigma}, \sigma^{2}=1, \sigma(\omega)=-\omega$.

$$
\left.N_{g, h}^{X, L_{O}}(\beta, \vec{d}):=\int_{\left[\overline{\mathcal{M}}_{g, h}\right.} X, L_{O}(\beta, \vec{d})\right]_{\text {vir }} 1
$$

[Katz-Liu, Li-Song '01]

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\mathrm{GW}_{g, h}^{X, L_{○}}:=\sum_{\vec{d}, \beta} N_{g, h}^{X, L_{O}}(\beta, \vec{d}) Q^{\beta} \vec{x}^{\vec{d}}
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Two stringy viewpoints on $\omega_{g h}^{M^{\beta}, \mathcal{K}}$
Take $B$ : spectral curves and the topological recursion


Two stringy viewpoints on $\omega_{g, h}^{M^{3}, \mathcal{K}}$

## Take B: spectral curves and the topological recursion

$\mathrm{CEO}_{0,1}\left[\mathcal{S}_{\bigcirc}\right]=\frac{\log Y(x)}{X}$
$\mathrm{CEO}_{0,2}\left[\mathcal{S}_{\bigcirc}\right]=B\left(X_{1}, X_{2}\right)$
$\mathrm{CEO}_{g, h}\left[\mathcal{S}_{\circ}\right]$
$=\sum_{\mathrm{d} X(p)=0} \operatorname{Res}_{p} K_{\mathrm{CEO}}\left[\mathcal{S}_{\mathrm{O}}\right]\left[\mathrm{CEO}_{g-1, h+1}+\sum_{g^{\prime}, h^{\prime}} \mathrm{CEO}_{g-g^{\prime}, h-h^{\prime}} \mathrm{CEO}_{g^{\prime}, h^{\prime}}\right]$

Two stringy viewpoints on $\omega_{g, h}^{M^{3}, \mathcal{K}}$

## Take B: spectral curves and the topological recursion

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## Two stringy viewpoints on $\omega_{g . h}^{M^{3}, \mathcal{K}}$

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$\left(\mathrm{CEO}_{g, 0}\left[\mathcal{S}_{\bigcirc}\right]=\mathrm{GW}_{g}(X)\right)$

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Want to: find viewpoint B for

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## Yeah but...

...this is just one (trivial) example!
Want to: find viewpoint $B$ for

- $S^{3} \longrightarrow M^{3}$ ? (in an arbitrary flat background)
- $\bigcirc \longrightarrow \mathcal{K}$ ?


## Part I: $S^{3} \longrightarrow M^{3}$

## $S^{3} \longrightarrow M^{3}$

Take $M^{3}$ a Clifford-Klein 3-manifold:

$$
\exists g \mid \operatorname{Ric}(g)>0 \Leftrightarrow M^{3} \simeq S^{3} / \Gamma, \Gamma \subset S O(4) .
$$

## Examples:

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\begin{aligned}
& \text { Type } A_{1}: \Gamma=\mathbb{Z} / 2 \Rightarrow M^{3} \simeq \mathbb{R} P^{3} \\
& \text { Type } E_{8}: \Gamma=\mathbb{I}_{120} \Rightarrow M^{3} \simeq \Sigma(2,3,5)
\end{aligned}
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(1) Find GW dual $X_{\Gamma}$, if any.
(2) Find spectral curve dual $\mathcal{S}_{\Gamma}$, if any.
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## The quest

## $U(N)$ CS on $S_{\text {「 }}$

large $N$ ?
A-model on...?


## $B$-model on...?

$S^{3} \longrightarrow M^{3}: A$-side

Geometric transition:

$S^{3} \longrightarrow M^{3}: A$-side

Geometric transition:


## $S^{3} \longrightarrow M^{3}: A$-side

Remarks [many things are well defined]:

- 「 action is fiberwise
(2) $X_{\Gamma}$ supports a $\mathbb{C}^{\star}$-Calabi-Yau action with compact fixed loci
(0) $\exists$ canonical Lagrangians $\mathcal{L}_{\Gamma} \subset X_{\Gamma}$ ( $\Rightarrow$ open (orbifold) GW invariants)
[Katz-Liu '01, AB-Cavalieri ' 10 ]
(0) orientifold constructions carry through $(S O(N) / S p(N))$
[Sinha-Vafa '00]
By (2) above, for $\Gamma \neq \mathbb{Z} / p \mathbb{Z}, X_{\Gamma}$ is non toric $\Rightarrow$ no Hori-lqbal-Vafa, no obvious mirror spectral curve


## Duality web

$U(N) C S$ on $S_{\Gamma}$ $\xrightarrow{\text { large } N ?} A$-model on $X_{\Gamma}$

$B$-model on...?
$\mathcal{N}=1\left(\mathcal{G}_{\Gamma}, \emptyset\right)$
SYM in $d=5$

## Duality web


$B$-model on
relativistic $\widehat{\mathcal{G}}_{\Gamma}$-Toda

## $\mathcal{G}_{\Gamma} / \widehat{\mathcal{G}_{\Gamma}}$-relativistic Toda

- Phase space: $\mathcal{P}_{\Gamma} \simeq\left(\mathbb{C}^{\star}\right)_{x}^{r_{\Gamma}} \times\left(\mathbb{C}^{\star}\right)_{y}^{r_{\Gamma}},\left\{x_{i}, y_{j}\right\}=C_{i j}^{\Gamma} x_{i} y_{j}$
- Dynamics: $\mathcal{C}:(\mathcal{P},\{\},) \rightarrow\left(\mathcal{G}_{\Gamma} / \mathcal{T}_{\mathcal{F}},\{,\}_{\text {DJov }}\right)$
$\left\{\mathcal{L}^{*} H_{i}, \mathcal{L}^{*} H_{j}\right\}=0, \quad H_{i} \in \mathcal{O}\left(\mathcal{G}_{\Gamma}\right)^{\mathcal{G}}$.
- Type A: non-periodic/periodic Ruijsenaars system
[Fock-Marshakov '97-'14, Williams '12, Kruglinskaya-Marshakov '14]


## $\mathcal{G}_{\Gamma} / \widehat{\mathcal{G}_{\Gamma}}$ relativistic Toda

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- Dynamics: $\mathcal{L}:(\mathcal{P},\{\},) \rightarrow\left(\mathcal{G}_{\Gamma} / \mathcal{T}_{\Gamma},\{,\}_{\text {DJov }}\right)$ $\left\{\mathcal{L}^{*} H_{i}, \mathcal{L}^{*} H_{j}\right\}=0, \quad H_{i} \in \mathcal{O}\left(\mathcal{G}_{\Gamma}\right)^{\mathcal{G}_{\Gamma}}$
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(a) Spectral differentials: $\left(\Omega_{1}, \Omega_{2}\right)=(\mathrm{d} \log \lambda, \mathrm{d} \log \mu)$.
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## $A D E_{6,7}$

Problem is trivial in type $A_{n}(\rho=\square)$, a back-of-the-envelope calculation for type $D_{n}\left(\rho=\left(\mathbf{2}_{v}\right)\right)$, and computable in reasonably short time on Mathematica for ( $E_{6}, 27$ ) (runtime: 30mins) and $\left(E_{7}, 56\right)$ (1/2 day).

## Todaг spectral curves: type $A$



## Todaг spectral curves: type $D$



## Todar spectral curves: type $E_{6}$



## Todar spectral curves: type $E_{7}$



- $\left(E_{8}, \rho=\mathfrak{e}_{8}\right)$ is completely unwieldy at face value, but it's related to the Poincaré sphere, it's immoral to leave anyone behind, and particularly frustrating when the most exceptional case is kept untreated.
- $\exists$ there's a semi-numerical way to break up the computation into a big number of smaller pieces of at most the size of the $E_{7}$ problem.
- Runtime grand total: 110 months, however code is easily "parallelisable".
With $N \simeq 75$ cores at once, this was reduced to about 1.5 months on a couple of small departmental clusters; see
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## Toda「 spectral curves: type $E_{8}$



## At the end of the day...

## Theorem (Borot-AB $\left.\left(A D E_{6,7}\right), \mathrm{AB}\left(E_{8}\right)\right)$

The B-model Gopakumar-Vafa duality holds in all genera for Clifford-Klein 3-manifolds in a reducible flat background - and for them alone.

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## SOTP:

- CS on Seifert manifolds reduces to a trigonometric (multi-)eigenvalue model
- $\exists \mathcal{P}_{\mathrm{CS}} \in \mathbb{C}[x, y] \mid \mathcal{P}_{\mathrm{CS}}\left(z, \omega_{0,1}^{S^{3} / \Gamma}(z)\right)=0$
[AB-Eynard-Mariño '11, Borot-Eynard '14, Borot-AB '15, AB '17]
- $\mathcal{P}_{\mathrm{CS}}=\left.\operatorname{det}(\mu-\mathcal{L}(\lambda))\right|_{u(t)} \Rightarrow \omega_{0,1}^{S^{3} / \Gamma}=\mathrm{CEO}_{0,1}$ [Todar]
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## SOTP:

- CS on Seifert manifolds reduces to a trigonometric (multi-)eigenvalue model
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## Application: mirrors of DZ Frobenius manifolds

Two meaningful operations:
(1) replace $\Omega_{1}=\mathrm{d} \log \lambda \rightarrow \mathrm{d} \lambda$ (Dubrovin's almost duality)
(2) restrict to degenerate leaf $\mathfrak{C} \rightarrow 0$

Combining both: bona-fide (conformal, with flat-unit) FM structure $\mathcal{F}_{\text {Toda }}^{\Gamma}$ on a suitable Hurwitz space.

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[Dubrovin-Zhang '96-'97]

## Theorem

For all simple Lie groups, we have
(1) $\mathcal{F}_{\text {Toda }}^{\Gamma} \simeq \mathrm{DZ}\left(\right.$ Weyl $\left._{\Gamma}\right)$
(2) choices of marked root in DZ correspond to choices of dual marked fundamental weights in $\mathcal{F}_{\text {Toda }}^{\Gamma}$
(3) for simply-laced $\Gamma$ and canonical choice of roots, $\mathcal{F}_{\text {Toda }}^{\Gamma} \simeq Q H_{\text {orb }}\left(\mathbb{P}_{\Gamma}^{1}\right)$
[Rossi '08, Zaslow '92]
(4) the higher genus GW potential of $\mathbb{P}_{\Gamma}^{1}$ coincides with the CEO higher genus free energies on $\mathcal{F}_{\text {Toda }}^{\Gamma}$.

Upon reversing Dubrovin's almost duality on the same leaf:

$$
\widehat{\mathcal{F}}_{\text {Toda }}^{\Gamma} \simeq Q H_{\text {orb }}\left(\left[\mathbb{C}^{2} / \Gamma\right]\right) \simeq Q H\left(\widehat{\mathbb{C}^{2} / \Gamma}\right)
$$

## Part II: $\bigcirc \longrightarrow \mathcal{K}$

## B-model for $\mathcal{K}$

The aim: Recall that for $\mathcal{K}=\bigcirc$, the full set of quantum invariants of the unknot were computed from the rational version of the topological recursion:

$$
\omega_{g, h}^{S^{3}, \bigcirc}=\mathrm{CEO}_{g, h}\left[\mathcal{S}_{\bigcirc}\right]
$$

Ideally, we'd like to have exactly the same for all knots.

## B-model for $\mathcal{K}$

## Four things we know/expect:

(1) spectral curves: 1-dimensional mirrors $\mathcal{S}_{\mathcal{K}}$ via the knot DGA and the augmentation polynomial of $\mathcal{K}$ :

$$
N_{X}(\Pi)=1-X-Y+Q X Y \longrightarrow \operatorname{Aug}_{\mathcal{K}} \in \mathbb{Z}[X, Y, Q]
$$

[Aganagic-Vafa '12, AENV '13, Ng '04]
(C) topological recursion: open/closed B-model on conic bundles $z w=P(X, Y) \in \mathbb{C}^{2} \times\left(\mathbb{C}^{*}\right)^{2}$ solved by Eynard-Orantin recursion;
[BKMP '07, Dijkgraaf-Vafa '08, Gu-Jockers-Klemm-Soroush '14]
(0) the closed sector: $\mathrm{CEO}_{g, 0}\left[\mathcal{S}_{\mathcal{K}}\right]=\mathrm{GW}_{g}(X)$ regardless of $\mathcal{K}$;
(4) symplectic invariance: if $\phi:\left(\left(\mathbb{C}^{\star}\right)^{2}, \omega\right) \rightarrow\left(\left(\mathbb{C}^{\star}\right)^{2}, \omega\right)$ is symplectic, then

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## The Main Conjecture (crude form)

Piecing everything together...

## Conjecture

There exists a set-theoretic correspondence $\mathcal{K} \rightarrow \phi_{\mathcal{K}} \in \operatorname{SCr}(2)$ (the symplectic Cremona group of the plane) such that

$$
\omega_{g, h}^{\mathcal{K}}=\mathrm{CEO}_{g, h}\left[\phi_{\mathcal{K}}^{*} N\left(\Pi_{x}\right)\right] \quad(\star)
$$

(cfr: mutations of LG potentials)
[Galkin-Usnich '10, Akhtar-Coates-Galkin-Kasprzyk '13]

## The symplectic Cremona group of the plane

Let $\omega=\mathrm{d} \log X \wedge \log Y$ be the standard holomorphic symplectic form on the 2-torus.

$$
\mathbb{C P}^{2} \quad: \quad \operatorname{Cr}(2)=\operatorname{Aut}_{\mathbb{C}}(\mathbb{C}(X, Y))
$$


[Usnich '08, Blanc '10]

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\uparrow & V \\
\left(\left(\mathbb{C}^{*}\right)^{2}, \omega\right) & : & \operatorname{SCr}(2)=\left\{\phi \in \operatorname{Cr}(2) \mid \phi^{*} \omega=\omega\right\}
\end{array}
$$

$$
\left\{\left(\mathbb{C}^{\star}\right)^{2}, \mathrm{SL}_{2}(\mathbb{Z}), \mathbb{Z} / 5=\langle P\rangle\right\}
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$$
\begin{aligned}
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\uparrow
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\mathrm{SL}_{2}(\mathbb{Z})=\left\langle C, \| C^{3}=I^{4}=1, I^{2} C=C I^{2}\right\rangle, P:(X, Y) \rightarrow\left(Y, \frac{Y}{X}+\frac{1}{X}\right)
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$\operatorname{SCr}(2) /\left(\mathbb{C}^{\star}\right)^{2} \simeq\left\langle C, I, P \mid C^{3}=I^{4}=P^{5}=1, I^{2} C=C I^{2}, I=P C P\right\rangle$
[Usnich '08, Blanc '10]

## The Main Conjecture (somewhat improved form)

## Conjecture

There exists a set-theoretic correspondence $\mathcal{K} \rightarrow \phi_{\mathcal{K}} \in \operatorname{SCr}(2)$ (the symplectic Cremona group of the plane) such that

$$
\omega_{g, h}^{\mathcal{K}}=\mathrm{CEO}_{g, h}\left[\phi_{\mathcal{K}}^{*} N\left(\Pi_{x}\right)\right] \quad(\star)
$$

Two (necessary) tweaks:
(1) conformal $\phi_{\mathcal{K}}: \phi_{\mathcal{K}}^{*} \omega=\alpha_{\mathcal{K}} \omega, \alpha_{k} \in \mathbb{Z}$
(2) symmetry action on the open string sector: $\Delta: \mathcal{S}_{\mathcal{K}} \rightarrow \mathcal{S}_{\mathcal{K}}$ (cfr: $\Gamma$-correspondences for 3-manifolds)

## What ( $\star$ ) would be

$\operatorname{SCr}(2) \xrightarrow{\boldsymbol{f}_{1}} T \triangleq\left\{\phi \in \operatorname{PL}\left(S^{1}\right) \mid \operatorname{Disc}(\phi)\right.$ dyadic partition, slopes $\left.\in 2^{\mathbb{Z}}\right\}$
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Ambiguities:

- Framing
- Unstable terms
- Classical terms


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(1) $\mathrm{CEO}_{g, 0}\left[\mathcal{S}_{\mathcal{K}}\right]=\mathrm{GW}_{g}(X)$;
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[AB-Eynard-Mariño '12]
(3) $\operatorname{Aug}_{\mathcal{K}}$ recovered from $\mathcal{S}_{\mathcal{K}}$;
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© all-qenus, coloured HOMFLY-PT from rational TR;
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